Course Name

Lecture 4: Geometric Series

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Questions?

Lecture Goal and Outcome

Goals:

- Recognise a geometric progression.
- Evaluate a geometric series.
- Determine if a geometric series is convergent or not.
- Use these concepts in some applications.

Learning Outcome: At the end of the lecture you will be able to solve problem like the following:

Problem

Determine for which real values of x the geometric series $3 + 6/x + 12/x^2 + 24/x^3 + \ldots$ converges, and for which value of x the series is equal to 5.

Why you should care

There are many problems in economics that can be solved with the aid of geometric series:

1. Value of an account with interest $p\%$ and regular payments of $\$a$.
2. The annuity problem
3. Mortage repaiments

Cumulative Revenue

This year a firm has an annual revenue of \$100 million that it expects to increase by 16%/year throughout the next decade. How large is its expected revenue in the tenth year, and what is the total revenue expected over the whole period?

Why you should care

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Retirment value

Your retirement account has \$5000 in it and earns 5% interest per year compounded monthly. Every month for the next 10 years you will deposit \$100 into the account. How much money will there be in the account at the end of those 10 years?

Saving account values

Kalle and Maria have just had a daughter, and have decided to open a bank account to save money for her university education. They would like to have \$100000 after 17 years. If the account pays 4% interest per year, compounded quarterly, and they make deposits at the end of every quarter, how large must each deposit be for them to reach their goal?

Why you should care

A construction firm wants to buy a building site and ha ste choice between three different payment schedules

1. Pay \$ 670 000 in cash
2. Pay \$120 000 a year for 8 years where the first installment is to be paid at once
3. Pay \$220 000 at once and then \$70 a year for 12 years where the first instalent is to be paid after one year.

Determine which schedule is least expensive is the interest rate is 11.5% and the firm has \$ 670 000 available in cash. What about if the interest rate is 12.5%? What about if the firm draws the money from an account that yields 4% interest rate, where the initial deposit is of \$ 670 000

Why you should care

Problem

What should an investor pay today for a perpetual (irredeemable) bond that pays a coupon of \$5 every six months if the appropriate rate to discount future cashflows is 4% per annum and the next coupon will be paid immediately? And if the first coupon is paid in a year?

Annuities

If n successive annual payments from a deposit are to be made, with the same amount \$a (due after every year). How much must be deposited in the account today in order to have enough savings to cover these n payments, given that the interest rate is p%/year?

You know from your previous courses that the answer to this problem is

$$K_0 = \frac{a}{I}\left(1 - \frac{1}{(1+I)^n}\right)$$

where $I = p/100$

Mortage repayments

A person borrows $\$D$ to be paid in n equal installments of $\$m$ at the end of each period T with compound interest rate in the period T of $p\%$

If we set $I = p/100$ as before we have that

$$D = \frac{m}{I}\left(1 - \frac{1}{(1+I)^n}\right)$$

if the first installment is due after the first period T has elapsed

$$D = \frac{m(1+I)}{I}\left(1 - \frac{1}{(1+I)^n}\right)$$

if the first installment is due immediately

Lecture Plan

- §11.4 Geometric Series
- (§11.5 Total present value)
- (§11.6 Mortgage repayments)

Section 1
Geometric progression

Geometric progression

If we invest \$100 at 10% interest compounded annually, then the future values in successive years are:

$$a_0 = 100, \quad a_1 = 100(1{,}1), \qquad a_2 = 100(1{,}1)^2,$$
$$a_3 = 100(1{,}1)^3, \ldots, \qquad a_n = 100(1{,}1)^n$$

In general, if we invest \$P at I% interest compounded annually, then the future values in successive years are:

$$a_0 = P, \quad a_1 = P(1 + \frac{I}{100}), \qquad a_2 = P(1 + \frac{I}{100})^2,$$
$$a_3 = P(1 + \frac{I}{100})^3, \ldots, \qquad a_n = P(1 + \frac{I}{100})^n$$

Note that

$$1 + \frac{I}{100} = \frac{a_1}{a_0} = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \ldots = \frac{a_{n+1}}{a_n}$$

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Definition

We say that a (possibly infinite) sequence of numbers $a_0, a_1, a_2, \ldots, a_n, \ldots$ is a geometric progression if there exists a constant r (called the (geometric) ratio) such that for all $n \geq 1$

$$r = \frac{a_{n+1}}{a_n}$$

In this case we can write $a_n = a_0 r^n, \quad n \geq 0,$

Example The formula $a_n = P(1 + \frac{I}{100})^n$ defines a geometric sequence with ratio $(1 + \frac{I}{100})$.

Example Which of the following sequences are geometric progressions? For those sequences that are of this type, write down their geometric ratios:

a) $1000, -100, 10, -1, \ldots$

b) $3, 6, 9, 12, \ldots$

c) $3, 6, 12, 24, \ldots$

d) $-1{,}3, -9, 27, \ldots$

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You have \$5000 to invest at 6%/year interest compounded monthly. How long will it take for your investment to grow to \$6000?

Active Learning

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Peer Instruction: Finding the Exponent n (3 Minutes)

From our monthly compounding rule on Page 14:

$$5000\left(1+\frac{0{,}06}{12}\right)^{n} = 6000 \implies (1{,}005)^{n} = 1{,}2$$

Clicker Check: Which algebraic setup correctly isolates the number of months n?

A) $n = \frac{1{,}2}{1{,}005}$

B) $n = \log(1{,}2) - \log(1{,}005)$

C) $n = \frac{\ln(1{,}2)}{\ln(1{,}005)}$

D) $n = (1{,}2)^{\frac{1}{1{,}005}}$

Action: Think for 60 seconds. On my count of three, raise 1, 2, 3, or 4 fingers. Then tell your neighbor why options A and D are common financial errors.

Section 2
Geometric Series

Geometric Series

> If you make a payment of \$100 at the end of every month into an account earning 3.6%/year, compounded monthly (this means that your investment is earning 3.6%/ 12 = 0.3%/month). What will be the value of the investment at the end of 1 year (12 months)?

The first payment will be compounded 11 times. So it will produce $100(1{,}003)^{11}$ at the end of the period.
The second payment will be compounded 10 times. It will produce $100(1{,}003)^{10}$ at the end of the period.

$\vdots$

The nth payment will be compounded $12 - n$ times. It will produce $100(1{,}003)^{12-n}$ at the end of the period.

1	2	3	4	5	6	7	8	9	10	11	12
p	$p \cdot r$	$p \cdot r^2$	$p \cdot r^3$	$p \cdot r^4$	$p \cdot r^5$	$p \cdot r^6$	$p \cdot r^7$	$p \cdot r^8$	$p \cdot r^9$	$p \cdot r^{10}$	$p \cdot r^{11}$
	p	$p \cdot r$	$p \cdot r^2$	$p \cdot r^3$	$p \cdot r^4$	$p \cdot r^5$	$p \cdot r^6$	$p \cdot r^7$	$p \cdot r^8$	$p \cdot r^9$	$p \cdot r^{10}$
		p	$p \cdot r$	$p \cdot r^2$	$p \cdot r^3$	$p \cdot r^4$	$p \cdot r^5$	$p \cdot r^6$	$p \cdot r^7$	$p \cdot r^8$	$p \cdot r^9$
			p	$p \cdot r$	$p \cdot r^2$	$p \cdot r^3$	$p \cdot r^4$	$p \cdot r^5$	$p \cdot r^6$	$p \cdot r^7$	$p \cdot r^8$
				p	$p \cdot r$	$p \cdot r^2$	$p \cdot r^3$	$p \cdot r^4$	$p \cdot r^5$	$p \cdot r^6$	$p \cdot r^7$
					p	$p \cdot r$	$p \cdot r^2$	$p \cdot r^3$	$p \cdot r^4$	$p \cdot r^5$	$p \cdot r^6$
						p	$p \cdot r$	$p \cdot r^2$	$p \cdot r^3$	$p \cdot r^4$	$p \cdot r^5$
							p	$p \cdot r$	$p \cdot r^2$	$p \cdot r^3$	$p \cdot r^4$
								p	$p \cdot r$	$p \cdot r^2$	$p \cdot r^3$
									p	$p \cdot r$	$p \cdot r^2$
										p	$p \cdot r$
											p

The first payment will be compounded 11 times. So it will produce $100(1{,}003)^{11}$ at the end of the period.
The second payment will be compounded 10 times. So it will produce $100(1{,}003)^{11}$ at the end of the period.

The 12th payment won't be compounded.

The value of the investment at the end of 12 months is

$$pr^0 + pr^1 + pr^2 + \ldots + pr^{11}$$

Geometric Series

Given a geometric progression

$$a_n = a_0 r^n$$

we are interested in calculate the finite geometric sums

$$s_n := \sum_{k=0}^{n} a_0 r^n = a_0(r^0 + r^1 + \ldots + r^n), \quad n \geq 0$$

Note that

$$\begin{aligned} rs_n &= a_0(r^1 + r^2 + \ldots + r^{n+1}) \\ &= a_0(r^0 + r^1 + r^2 + \ldots + r^n + r^{n+1} - r^0) = s_n - a_0(1 - r^{n+1}), \end{aligned}$$

This implies that (if $r \neq 1$)

$$a_0(1 - r^{n+1}) = (1 - r)s_n \Leftrightarrow s_n = a_0 \frac{1 - r^{n+1}}{1 - r}$$

Finite Geometric Series

$$\sum_{k=0}^{n} a_0 r^k = \begin{cases} a_0 \frac{1-r^{n+1}}{1-r} & \text{if } r \neq 1 \\ a_0(n+1) & \text{if } r = 1 \end{cases} \qquad n \geq 0$$

Example In the previous example, we reduce the problem to calculate, for $p = 100$, $r = 1{,}003$, $n = 11$

$$pr^0 + pr^1 + pr^2 + \ldots + pr^{11} = p\frac{1 - r^{12}}{1 - r} = 100\frac{1 - (1{,}003)^{12}}{-0{,}003}$$
$$\approx 1220$$

Example

If you make a payment of \$100 at the end of every month into an account earning 3.6%/year, compounded monthly. This means that your investment is earning 3.6%/ 12 = 0.3%/month. How many months will have to pass for the value of the investment to be at least \$100000?

Now, what we don't know is the number of months, say n. We would like to find n such that

$$pr^0 + pr^1 + pr^2 + \ldots + pr^{n-1} = 100\frac{(1{,}003)^n - 1}{0{,}003} \geq 100000$$

which is equivalent to find a natural number n such that

$$(1{,}003)^n \geq 3 + 1 \Leftrightarrow n \geq \frac{\ln(4)}{\ln 1{,}003} \approx 462{,}7$$

So we need to wait at least 463 months that is $\approx$ 38 years!

The annuity problem

Problem

You have \$13000 in an account that gives you 2% a year. You buy a car and you are given the following alternatives:

- You pay \$13000 now;
- You pay pay \$13000 in 10 years with an interest rate of 6% a year.
- You pay \$2000 right now, and the rest in 9 payments of \$1300 due at the beginning of the year. First payment is due next year

What do you choose?

Active Learning 2: Conceptually Ranking Options

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Think-Pair-Share: The Car Buying Dilemma (5 Minutes)

Look at the three choices on your screen for purchasing the $\$13,000$ vehicle (Slide 21). Let's build our economic intuition before writing down big sigma notation sums.

Your Task:

1. **Think (1 min):** Without computing any fractions, why might paying $\$13,000$ in 10 years at 6% interest (Option 2) be cheaper or more expensive than paying cash up front if your bank account yields 2%?
2. **Pair (2 mins):** Compare preferences with your bänkkamrat. Did you pick the same option?
3. **Share (2 mins):** Let's run the actual math to check our financial baseline!

Annuities

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If *n* successive annual payments from a deposit are to be made, with the same amount \$a (due after every year). How much must be deposited in the account today in order to have enough savings to cover these *n* payments, given that the interest rate is p%/year?

You know from your previous courses that the answer to this problem is

$$K_0 = \frac{a}{I}\left(1 - \frac{1}{(1+I)^n}\right)$$

where $I = p/100$

Why is that?

In order to have the amount after

- one year, we must deposit an amount x_1

  $$x_1(1+I) = a \Leftrightarrow x_1 = a(1+I)^{-1}$$

- two year, we must deposit an amount x_2

  $$x_2(1+I)^2 = a \Leftrightarrow x_2 = a(1+I)^{-2}$$

- n years, we must deposit an amount x_n

  $$x_n(1+I)^n = a \Leftrightarrow x_n = a(1+I)^{-n}$$

So the total amount deposit today has to be $a\sum_{j=1}^{n}\left((1+I)^{-1}\right)^j$

N.b. An **annuity** is a sequence of equal payments made at fixed periods of time over some time span.

Geometric series

$$\sum_{k=l}^{n} a_0 r^k = r^l \sum_{k=0}^{n-l} a_0 r^k = \begin{cases} a_0 r^l \frac{1-r^{n-l+1}}{1-r} & \text{if } r \neq 1 \\ a_0(n-l+1) & \text{if } r = 1 \end{cases}$$

In the example above

$$a \sum_{j=1}^{n} \left((1+I)^{-1}\right)^j = a(1+I)^{-1} \sum_{j=0}^{n-1} \left((1+I)^{-1}\right)^j$$
$$= \frac{a}{I}\left(1 - \frac{1}{(1+I)^n}\right)$$

This formula gives the present value of n future claims of \$a/each.

In the same situation as above, how much be deposited in the account today in order to cover an infinite number of payments? To find the answer, we need to calculate

$$\begin{aligned}\lim_{n\to\infty} a\sum_{j=1}^{n}\left((1+I)^{-1}\right)^{j} &= \lim_{n\to\infty}\frac{a}{I}\left(1-\frac{1}{(1+I)^{n}}\right)\\ &= \frac{a}{I}\left(1-\lim_{n\to\infty}\frac{1}{(1+I)^{n}}\right)\\ &= \frac{a}{I},\end{aligned}$$

Depositing $\frac{a}{I}$ we make sure that we can withdraw an amount $\$a$/year forever.

Mortage repayments

A person borrows $\$D$ to be paid in n equal installments of $\$a$ at the end of each period T with compound interest rate in the period T of $p\%$

If we set $I = p/100$ as before we have that

$$D = \frac{a}{I}\left(1 - \frac{1}{(1+I)^n}\right)$$

if the first installment is due after the first period T has elapsed

$$D = \frac{a(1+I)}{I}\left(1 - \frac{1}{(1+I)^n}\right)$$

if the first installment is due immediately

Section 3
Convergent geometric series

Infinite geometric series - motivation

The formula for consols

A consol is an instrument which has an infinite number of coupon payments at the specified coupon rate. We have a consol with fixed coupon rate c, yielding y and face value F. How do we set the price for the consol?

We need to compute

$$P = cF\left(\sum_{i=1}^{\infty} \frac{1}{(1+y)^i}\right)$$

Convergent Series

We say that the geometric series $\sum_{j=l}^{n} r^k$ converges if $\lim_{n\to\infty} \sum_{j=l}^{n} r^k$ exists as a real number. If so, we denote

$$\sum_{k=l}^{\infty} r^k := \lim_{n\to\infty} \sum_{j=l}^{n} r^k$$

Notice that for $r \neq 1$

$$\lim_{n\to\infty} \sum_{k=l}^{n} r^k = \lim_{n\to\infty} r^l \frac{1 - r^{n-l+1}}{1-r} = \begin{cases} \frac{r^l}{1-r} & \text{if } |r| < 1 \\ \infty & \text{if } |r| > 1 \end{cases}$$

Convergent Series

The geometric series $\sum_{j=l}^{n} r^k$ converges if and only if $|r| < 1$.

Intuition Behind Infinite Convergence

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Group Challenge: The Perpetual Cash Machine (4 Minutes)

A financial asset promises to pay you \$100 this year, \$50 next year, \$25 the year after, halving your payout forever ($r = 0{,}5$). Another asset pays \$100 every year without ever decreasing ($r = 1{,}0$).

Discuss with your group:

- Why does the total lifetime payout of the first asset stop at a finite ceiling, while the second one blows up to infinity?
- Connect this to the condition on Slide 29: Why must $|r| < 1$ for an infinite series to converge to a real number?

Example

Problem

What should an investor pay today for a perpetual (irredeemable) bond that pays a coupon of \$5 every six months if the appropriate rate to discount future cashflows is 4% per annum and the next coupon will be paid immediately? And if the first coupon is paid in a year?

The total yielding of the bond will be

$$\sum_{n=0}^{\infty} 5(1{,}002)^{-n}$$

if the first coupon is paid immediately and

$$\sum_{n=2}^{\infty} 5(1{,}002)^{-n}$$

Section 4
Applications

Applications

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A construction firm wants to buy a building site and ha ste choice between three different payment schedules

1. Pay \$ 670 000 in cash
2. Pay \$120 000 a year for 8 years where the first installment is to be paid at once
3. Pay \$220 000 at once and then \$70000 a year for 12 years where the first instalent is to be paid after one year.

Determine which schedule is least expensive is the interest rate is 11.5% and the firm has \$ 670 000 available in cash. What about if the interest rate is 12.5%?

Solution: Evaluating the Three Payment Schedules

The Economist's Objective

To determine the least expensive schedule, we must find the **Total Present Value (PV)** of each option discounted at the market interest rate *I*.

Case A: Interest Rate is 11,5% ($I = 0{,}115$)

- **Schedule 1 (Cash Upfront):** No future discounting needed.

$$\text{PV}_1 = \$\mathbf{670,000}$$

- **Schedule 2 (8 annual payments of \$120,000, first due immediately):** This is an annuity due.

$$\text{PV}_2 = 120,000 + \sum_{k=1}^{7} \frac{120,000}{(1{,}115)^k} = 120,000 + \frac{120,000}{0{,}115}\left(1 - \frac{1}{(1{,}115)^7}\right)$$

Solution: Schedule 3 & Final Decision at $11{,}5\%$

Schedule 3 (\$220,000 now + 12 annual payments of \$70,000 starting in Year 1):

- This is an immediate cash outflow plus a standard 12-year ordinary annuity:

$$\mathrm{PV}_3 = 220,000 + \sum_{k=1}^{12} \frac{70,000}{(1{,}115)^k} = 220,000 + \frac{70,000}{0{,}115}\left(1 - \frac{1}{(1{,}115)^{12}}\right)$$

$$\mathrm{PV}_3 = 220,000 + 441,747 = \$\mathbf{661,747}$$

Decision Matrix at $11{,}5\%$

Comparing the Present Values:

$$\mathrm{PV}_3\ (\$661{,}747) < \mathrm{PV}_1\ (\$670{,}000) < \mathrm{PV}_2\ (\$675{,}273)$$

Solution: What happens if the Interest Rate rises to 12,5%?

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Economic Intuition Check

When the interest rate (discount rate) increases, the present value of *future* cash outlays **decreases**. Schedules with delayed payments should become relatively more attractive!

Recalculating with $I = 0{,}125$:

- **PV_1 (Cash):** Remains completely unchanged = $\$\mathbf{670,000}$
- **PV_2 (8 payments):**

$$\mathrm{PV}_2 = 120,000 + \frac{120,000}{0{,}125}\left(1 - \frac{1}{(1{,}125)^7}\right) = \$\mathbf{656,081}$$

- **PV_3 (12 payments):**

$$\mathrm{PV}_3 = 220,000 + \frac{70,000}{0{,}125}\left(1 - \frac{1}{(1{,}125)^{12}}\right) = \$\mathbf{638,623}$$

Applications

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You wish to establish a trust fund from which your daughter can withdraw \$1,000 every six months for 10 years. At the end of that time, she will get the remaining money in the trust, which you would like to be \$50,000. The trust will be invested at 7%/year compounded every six months. How large should the trust be?

A person borrows \$50 000 at the beginning of a year and is supposed to pay it off in 5 equal instalments at the end of each year, with interest at 15% compounding annually. Find the annual payment.

If \$a is amount of each repayment, the present value is

$$\sum_{k=1}^{5} \frac{a}{(1{,}15)^k} = a(1{,}15)^{-1} \sum_{k=0}^{4} \left(\frac{1}{1{,}15}\right)^k$$
$$= a(1{,}15)^{-1} \frac{1-(1{,}15)^{-5}}{1-(1{,}15)^{-1}} = \frac{a}{0{,}15}(1-(1{,}15)^{-5})$$

This quantity has to be equal to 50 000, so

$$\frac{a}{0{,}15}(1-(1{,}15)^{-5}) = 50000 \Leftrightarrow a \approx 14915{,}78$$

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> Suppose the loan of \$50 000 is to be repaid by paying \$20 000, which covers both interest and the principal repayment, at the end of each of the coming years, until the loan is fully paid off. When is the loan paid off if the annual rate is 15%?

In this case we need to find n such that

$$\sum_{k=1}^{n} \frac{2 \times 10^4}{(1{,}15)^k} = \frac{2 \times 10^4}{0{,}15}(1 - (1{,}15)^{-n}) \geq 5 \times 10^4,$$

This is equivalent to solve

$$1-(1{,}15)^{-n} \geq \frac{5 \times 15}{2 \times 100} \Leftrightarrow \ln(1 - \frac{3}{8}) \geq -n \ln(1{,}15)$$

$$\Leftrightarrow n \geq \frac{\ln(\frac{8}{5})}{\ln(1, 15)} \approx 3{,}36$$

Thank you for your attention!