MM3001

Lecture 5: Limits and Derivatives

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Review & Clarifications

Opening the Floor for Questions

Before we begin today's material on Limits and Derivatives, what questions do you have regarding **Geometric Sums, Annuities, or Present Value Calculations**?

60-Second Flash Recall: Warming Up Your Financial Brain

To help recall last week's core concepts while you look over your notes for questions:

- What is the critical mathematical condition required for an infinite geometric series to converge to a finite number?
- If a multi-year cash flow stream features a growth ratio of $r = 1{,}05$, can we calculate its infinite horizon present value? Why or why not?

Average vs. Instantaneous Change

Think-Pair-Share: The Cost of a Production Spike (3 Minutes)

A Swedish production plant finds that increasing its daily output from 100 to 110 units raises total cost by $5,000$ kr (500 kr per unit on average).

Discuss with your bänkkamrat:

- Does this average tell the factory manager the exact cost of producing just the **101st unit**? Why or why not?
- If the cost function is curved, how can we mathematically narrow our measurement window down to a single instantaneous point?
- *Hint:* Today we will look at how **Limits** allow us to shrink this interval down to zero!

Lecture Goal and Outcome

Goal: Introduction to limits and derivatives (If you have taken it, you should have seen some of this material in your Matte 3 class)

Learning Outcome: After today you will be able to solve problems like the followings:

Problem

If the total saving of a country is a function $S(Y)$ of the national product Y, then $S'(Y)$ is called the marginal propensity to save, or MPS. Find the MPS for the following functions:

- $S(Y) = S_0 + sY$
- $S(Y) = 100 + 0{,}1Y + 0{,}0002Y^2$

Lecture Plan

- Limits (6.5)
- Derivatives (6.1-6.3, 6.6-6.11)

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Attention

We use a different order than in the book.

Section 1
Limits

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Consider the function

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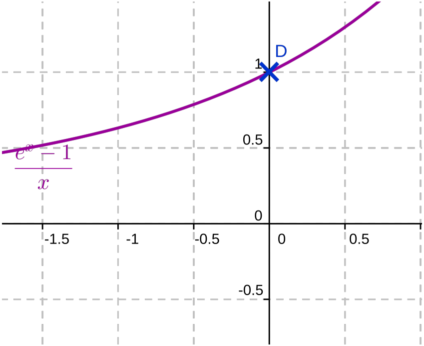

x	$f(x) \simeq$
-1	0.632
-0.1	0.956
-0.001	0.999
0	?
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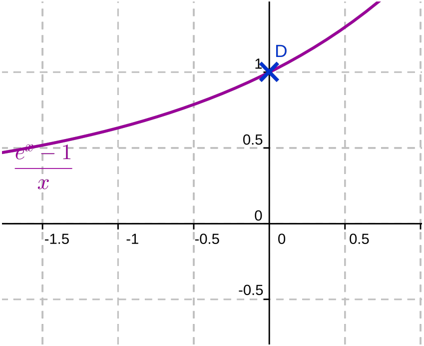

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$$\lim_{x \to 0} f(x) = 1$$

Definition of limit

Definition (p. 277)

Let f be a function over an interval I, and let a be in I. If for every $\epsilon > 0$ we have that that there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $x \in I$ is such that $|x - a| < \delta$, we say that f has limit L when x goes to a:

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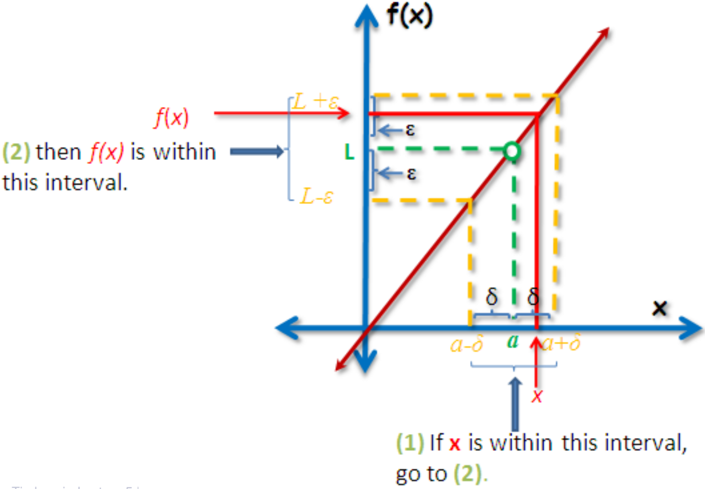
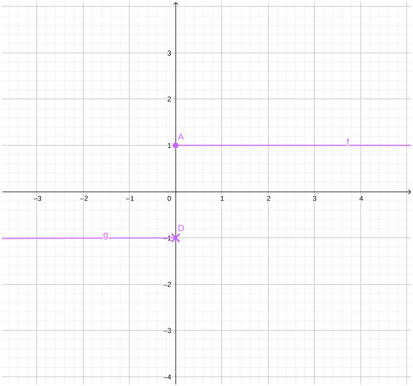
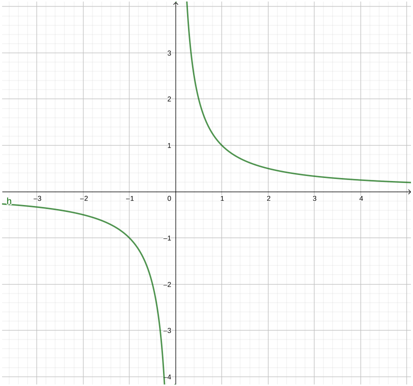
What does this mean?

Heuristically this means that when x get close to a $f(x)$ get close to L



Sometimes it does not exist





Rule for limits (to know!!!)

Let f and g two function on I and let

$$A := \lim_{x \to a} f(x), \quad B := \lim_{x \to a} g(x)$$

Then

- $\lim_{x \to a}(f(x) \pm g(x)) = A \pm B$
- $\lim_{x \to a}(f(x) \cdot g(x)) = A \cdot B$
- If c is a real number $\lim_{x \to A} cf(x) = cA$
- If $B \neq 0$, $\lim_{x \to a}(f(x)/g(x)) = A/B$
- If $f(x) < g(x)$ (or $f(x) \leq g(x)$) for every $x \in I$, then $A \leq B$

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There is also a rule for the composition:
Suppose that $\lim_{x \to a} f(x) = A$ and that $\lim_{y \to A} g(y) = B$, then

$$\lim_{x \to a} g \circ f(x) = B$$

Exercises

- $\lim_{x\to 2} \frac{x^2-3x+2}{x-1}$
- $\lim_{x\to 1} \frac{x^2-3x+2}{x-1}$
- $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$

Resolving the "0/0"

Peer Instruction: The Limit of Instantaneous Revenue Growth

Suppose the change in a startup's revenue over an abbreviated window of time h is modeled by the expression we saw on Slide 7:

$$f(h) = \frac{e^h - 1}{h}$$

We want to find the true baseline growth rate at the exact launch moment where $h \to 0$.

Concept Check: Why can't we just plug in $h = 0$ directly?

A) Because $e^0 = 0$, making the top zero.

B) It creates an undefined $\frac{0}{0}$ fraction.

C) Exponential functions don't have limits.

D) The limit is obviously ∞.

Questions?

Section 2
Derivatives

Tangents

Problem

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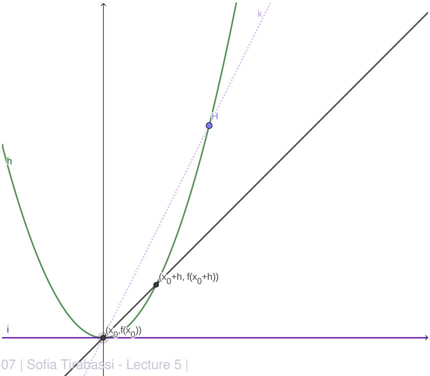

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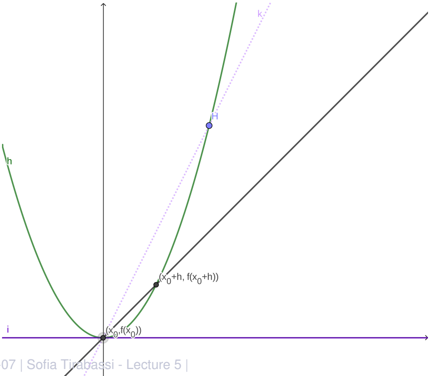

The slope of the secant is

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

Slope of the tangent

We have found that the slope of the secant line to the graph passing for $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h)$ is

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Definition

If the $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists, we denote it be $f'(x_0)$ and we call it the derivative of f at x_0. We say that f is differentiable at x_0. If f is differentiable at any point of an interval I we say that f is differentiable on I.

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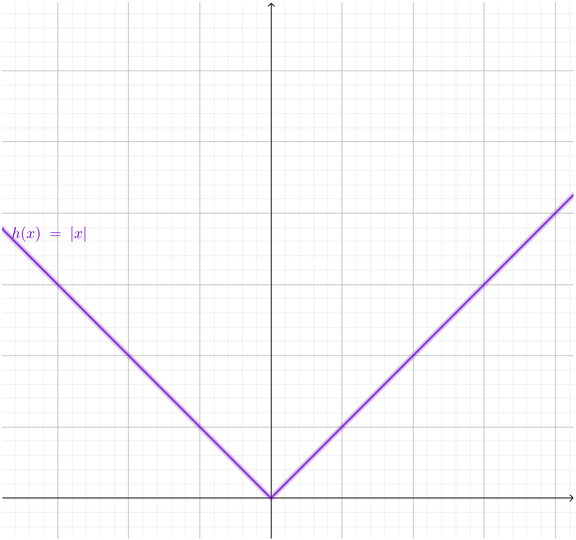
There are other symbols to denote the derivative of f at x_0:

$$D[f](x_0),\ \frac{df}{dx}(x_0),\ \dot{f}(x_0),\ \frac{df}{dx}_{|x=x_0},\dots$$

It can not exist



Exercise

Find the tangent line to $y = x^2 + 2x - 1$ in $(2, 7)$.

Interpretation of $f'(x)$

In general $f'(x)$ represents the rate of change of f around x.

Example

If $V(t)$ represent the value of a portfolio at time t then $V'(t)$ represent how that value is changing. In particular if $V'(t) > 0$, then the value is going up.

Increasing/Decreasing functions

- A function $f(x)$ is increasing on an interval I iff for every $x \in I$ we have that $f'(x) \geq 0$
- A function $f(x)$ is strictly increasing on an interval I if for every $x \in I$ we have that $f'(x) > 0$
- A function $f(x)$ is decreasing on an interval I iff for every $x \in I$ we have that $f'(x) \leq 0$.
- A function $f(x)$ is strictly decreasing on an interval I if for every $x \in I$ we have that $f'(x) < 0$.

Derivatives of basic functions

$f(x)$	$f'(x)$
c	0
x^q	qx^{q-1}
e^x	e^x
$\ln x$	$\frac{1}{x}$

Rules for derivation

- $D[f+g](x) = f'(x) + g'(x)$
- $D[f \cdot g](x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
- $D[1/g](x) = -\frac{g'(x)}{(g(x))^2}$
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Attention

You need to know how to use these rules for the exam. Practice!

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Important remark: Several of these rules have additional conditions, such as the derivatives of each of the function involved must exist, or, if you divide by $g(x)$ this cannot be 0...

Unpacking the Chain Rule Structure

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Inner vs. Outer Economic Functions

A macroeconomist models national consumption C as a function of disposable income, which itself depends on the global oil tax rate t:

$$C(t) = \ln(2t^2 + 5)$$

Identify the components for the Chain Rule $D[f(g(t))] = f'(g(t)) \cdot g'(t)$:

- What is the **outer function** $f(u)$, and what is its derivative $f'(u)$?
- What is the **inner function** $g(t)$, and what is its derivative $g'(t)$?

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Interactive Solution Matrix

Outer: $f(u) = \ln(u) \implies f'(u) = \frac{1}{u}$
Inner: $g(t) = 2t^2 + 5 \implies g'(t) = 4t$
Combined: $C'(t) = \frac{1}{2t^2+5} \cdot 4t = \frac{4t}{2t^2+5}$

Questions?

The initial problem

Problem

If the total saving of a country is a function $S(Y)$ of the national product Y, then $S'(Y)$ is called the marginal propensity to save, or MPS. Find the MPS for the following functions:

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Further examples

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- Compute the derivative of a^x
- Compute the derivative of $\log_a(x)$
- (part of exam question) Compute the derivative of

$$\sqrt{e^{\sqrt{x^2+1}}}$$

- Compute the derivative of $\ln|x|$, when it exists
- Compute the derivative of $(2x-1)(e^{x^2}+2x-1)^{25}$
- Determine on which interval the function $e^x(x^2-2x+1)$ is increasing/decreasing. Sketch the graph.

Exam Strategy Workshop

Group Challenge: Attacking the Complex Exam Derivative (5 Minutes)

Look at the core exam problem listed on Slide 25:

$$y = \sqrt{e^{\sqrt{x^2+1}}} = \left(e^{(x^2+1)^{1/2}}\right)^{1/2} = e^{\frac{1}{2}\sqrt{x^2+1}}$$

Work with your team to sequence your differentiation steps:

1. **Step 1:** What rule from Slide 21 applies to the outermost base e layer?
2. **Step 2:** When you apply the Chain Rule to the exponent $\frac{1}{2}(x^2+1)^{1/2}$, what is the final internal multiplier generated by the x^2+1 core?
3. **Step 3:** Combine your expressions to state the clean derivative y'.

Questions?

Higher order derivatives

Observe that the derivative of a function its itself a function and so we can keep on deriving we get

$$f''(x) = \frac{d^2 f}{dx^2}(x)$$
$$f'''(x) = \frac{d^3 f}{dx^3}(x)$$

In general we have

$$f^{(n)}(x) = \frac{d^n f}{dx^n}$$

Thank you for your attention!