MM3001

Lecture 6 : Implicit derivations and Taylor's formula

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Review & Clarifications: Lecture 5

Opening the Floor for Clarifications

Before we dive into implicit functions today, what questions do you have regarding last lecture's material on **Limits, Differentiation Rules, or the Chain Rule**?

60-Second Flash Recall: Warming Up Your Derivative Brain

To help recall core mechanics while you look through your notes for questions:

- What is the core rule when differentiating a nested composite function like $f(g(x))$?
- If $y = e^{3x^2}$, what is the final internal multiplier generated by the exponent?

Take a moment to check your notes from last week and ask any

Lecture Goal and Outcome

Goal: Overview of implicit derivation and polynomial approximation.
Learning Outcome: At the end of today lecture you will be able to

solve a problem like the following:

Problem

Suppose that a commodity is sold at price P with a taxation of τ. Suppose that the demand and supply functions are linear, that is

$$D = a - b(P + \tau), \quad S = \alpha + \beta P,$$

Then the equilibrium price is determined by equating the supply to the demand:

$$a - b(P + \tau) = \alpha + \beta P$$

Compute $\frac{dP}{d\tau}$ What is its sign? What is the sign of $\frac{d}{d\tau}(P + \tau)$

Outcome 2

After today you will understand the rule of 70:

Rule of 70

Given an interest rate of $p\%$ the doubling time is approximately $70/p$ years.

Why you should care

- Implicit derivation is a really important technique in economics, as sometimes it is impossible to define a quantity Y directly as a function of another. But still one would want to examine how the two quantities are correlated.
- As you are going to deal with real data and real functions, it will be important to approximate them, otherwise some problems cannot be solved.
- Whenever you approximate something it is really important to have an idea of the magnitude of your error. Taylor's formula provide just that for the approximation we will learn today.

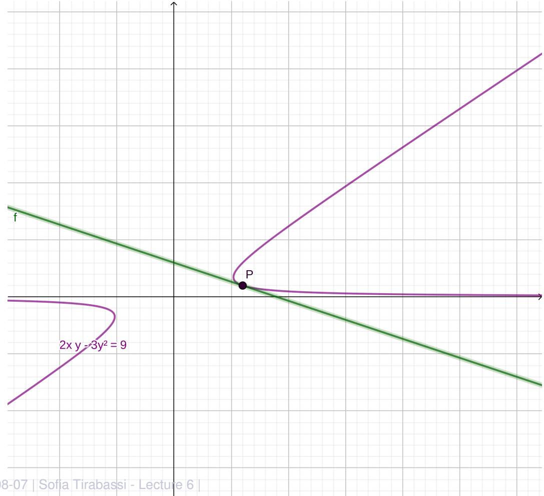
Lecture Plan

- Implicit derivation (7.1) - usually an exam question
- Linear approximation (7.4)
- Polynomial approximation (7.5)- has always been an exam question.
- Taylor's Formula (7.6)

Section 1
Implicit Derivation

Example

Consider the plane curve $2xy - 3y^2 = 9$. We want to compute the slope of the tangent to the curve in the point $(6, 1)$.



The general method

We have an equation $f(x, y) = 0$ and consider $y = y(x)$ we want to determine $y'(x)$ in a point (x_0, y_0).

- We differentiate each side of $f(x, y) = 0$ with respect to x, considering y as a function of x. We get
$$\frac{d}{dx} f(x, y(x)) = 0,$$
- We set $x = x_0$ and $y = y_0$ and we get an equation in which the only unknown is $y'(x_0)$
- We solve the equation.

If we want to compute $y''(x)$, we repeat the process:

- We compute
$$\frac{d^2}{dx^2} f(x, y(x)) = 0$$
- We set in $x = x_0$, $y = y_0$ and $y'(x) = y'(x_0)$ which we have previously computed.
- We solve for $y''(x_0)$

The Inital problem

Think-Pair-Share: Comparative Statics (5 Minutes)

Let us solve our lecture goal from Slide 2. The equilibrium price under a tax rate τ is implicitly bound by equating supply and demand:

$$a - b(P + \tau) = \alpha + \beta P$$

Treat P as an implicit function of τ ($P = P(\tau)$). All other letters are constants.

Your Task:

1. **Think (2 mins):** Differentiate both sides with respect to τ. Remember that $\frac{d}{d\tau}[P] = \frac{dP}{d\tau}$ and $\frac{d}{d\tau}[\tau] = 1$.
2. **Pair (2 mins):** Isolate $\frac{dP}{d\tau}$ and determine its economic sign given that parameters $b, \beta > 0$.
3. **Share (1 min):** Does a tax hike increase or decrease the equilibrium price P?

Further Examples

Determine the slope of the tangent of the unit circle at any point (but not (1,0) and (-1,0))

Further Examples

From an old exam

Consider the following curve

$$y^3e^x + x^2e^y + y^3 - 2y = C,$$

for some real number C.

1. Suppose that the curves passes for $(1,1)$, determine C,
2. Determine the slope of the curve (that is the slope of its tangent) in the point $(1,1)$.
3. Consider y as a function of x. Is this increasing around $x = 1$?
4. Determine the equation of the tangent line in the point $(1,1)$

Further Examples

From an old exam

Consider the curve

$$y^2x^2 + \frac{x}{\sqrt{y}} = 6$$

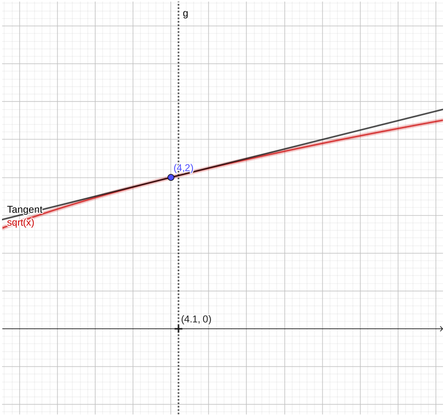
1. Compute the slope of the curve in $(2, 1)$.
2. Consider y as a function of x, compute $y''(2)$.

Questions?

Section 2
Linear approximation

Example

We want to approximate $\sqrt{4{,}1}$.
Idea: We find $y = t(x)$, the equation of the tangent to the graph of $y = \sqrt{x}$ in the point $x = 4$. Then $t(4{,}1)$ approximates $\sqrt{4{,}1}$.



Unlocking the Rule of 70

Group Challenge: The Linearization of Growth (4 Minutes)

Previously, we stated the Rule of 70: Doubling time $\approx 70/p$. An asset doubling at interest rate $r = p/100$ solves:
$(1+r)^t = 2 \implies t = \frac{\ln(2)}{\ln(1+r)}$.

Work with your bänkkamrat to complete the proof:

1. Find the linear approximation of $f(r) = \ln(1+r)$ centered at $a = 0$.
2. *Hint:* $f(0) = \ln(1) = 0$, and $f'(r) = \frac{1}{1+r} \implies f'(0) = 1$.
3. Substitute $\ln(1+r) \approx r$ into the time formula: $t \approx \frac{\ln(2)}{r}$.
4. Given that $\ln(2) \approx 0{,}693$, write this in terms of percentage p $(r = p/100)$.

Questions?

Section 3
Polynomial approximation

Quadratic approximation

Instead of approximate a function with a line, we approximate it with a quadratic function. That is we want to find

$$p(x) = A + B(x-a) + C(x-a)^2$$

such that $p(x)$ is really close to $f(x)$ when x is near a. We require the following conditions

- $f(a) = p(a)$
- $f'(a) = p'(a)$
- $f''(a) = p''(a)$

We get a unique solution

$$\boxed{p(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2}$$

Examples

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- Find the quadratic approximation of $\sqrt{4{,}1}$
- Find the quadratic approximation of $\ln(1 + \frac{p}{100})$

Higher-order approximations

The Taylor Polynomial

Given a function f differentiable n times at a point a, we have that the degree n Taylor polynomial associated to f in a is:

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Examples

- Compute the Taylor polynomial of degree 3 of $f(x) = e^x$ around $x = 1$. How do you think the one of degree 50 looks like?
- (Exam question) Compute $p(x)$ the Taylor polynomial of degree 3 of the function $f(x) = x^2 e^x$ around $x = 1$. What is the value of $p(1{,}1)$?
- (Exam question) Compute $p(x)$ the Taylor polynomial of degree 3 of the function $f(x) = \ln(1 + e^x)$ around $x = 0$. What is the value of $p(0{,}1)$?
- (Exam question) Compute $p(x)$ the Taylor polynomial of degree 3 of the function $f(x) = \ln(x^2 + 2x + 1)$ around $x = 1$.

Factorial Diagnostic Workshop

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Peer Instruction: Tracking Taylor Coefficients (3 Minutes)

The general Taylor polynomial of degree n (Slide 22) scales its terms by $\frac{f^{(n)}(a)}{n!}$. Let's avoid common test traps.

Concept Check: Suppose a function's 3rd derivative evaluated at $a = 1$ is $f^{(3)}(1) = 12$. What is the exact coefficient assigned to the $(x - 1)^3$ term in the degree 3 Taylor polynomial?

A) 12

B) 4

C) 2

D) 6

Action: On my count of three, raise fingers matching your choice. Discuss with your neighbor why forgetting that $3! = 3 \times 2 \times 1 = 6$ costs easy points on exams!

Questions?

Section 4
Taylor's Formula

Question

How good are our approximations?

Taylor's Formula

Suppose that f is a function differentiable $n+1$ times in an interval I containing the point a. Let $p(x)$ be the Taylor polynomial of degree n of $f(x)$ in a. For every $x \in I$ there is a $z \in I$ such that

$$|f(x) - p(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1} \right|$$

Example

Compute $e^{1/10}$ with two correct decimals. Suppose that $p(x)$ is the Taylor polynomial of degree n centered in 0 of e^x. We want to compute $p(0{,}1)$ in such a way that

$$|p(0{,}1) - e^{0{,}1}| < 0{,}005$$

Taylor's formula gives us that

$$|p(0{,}1) - e^{0{,}1}| = \left| \frac{f^{(n+1)}(z)}{(n+1)!}(0{,}1)^{n+1} \right|$$

for some z in an interval containing 0 and 0,1. Let us take $I = [0, 0{,}1]$. We have

$$\left| \frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1} \right| = \left| \frac{e^z}{(n+1)!}(0{,}1)^{n+1} \right| \leq \frac{e^{0{,}1}}{10^{n+1}(n+1)!} \leq \frac{2}{10^{n+1}(n+1)!}$$

Questions?

Thank you for your attention!