MM3001

Lecture 7 : More on limits, continuity, and introduction to integration

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Review & Clarifications:

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Opening the Floor for Clarifications

Before we begin today's material on Continuity and Integration, what questions do you have regarding last lecture's material on **Implicit Differentiation or Taylor Polynomials**?

60-Second Flash Recall: Warming Up Your Mathematical Brain

To help recall core mechanics while you look through your notes for questions:

- When differentiating an expression like $y^3 + x^2 y = 5$ implicitly with respect to x, what rule must be applied to the $x^2 y$ term?
- What notation change separates a standard quadratic function from a degree 2 Taylor approximation $P_2(x)$?

Take a moment to check your notes from last week and ask any clarifications!

Lecture Goal and Outcome

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Goal: Understand continuous functions and their properties.
Compute limits with l'Hôpital's rule.

Learning Outcome: At the end of today lecture you will be able to

- decide if a function is continuous by looking at its law
- Compute the following

$$\lim_{x \to 1} \frac{\ln(x) - x + 1}{x - 1}$$

Lecture Plan

- Continuous functions (7.8)
- Limits at infinty (7.9)
- L'Hôpital's rule (7.12)
- Indefinite Integrals (10.1)

Section 1
Continuous functions

Continuous function

Definition

We say that a function f defined on an interval I is continuous at $a \in I$ if the following two conditions hold:

- the limit $\lim_{x \to a} f(x)$ exists.
- we have that $\lim_{x \to a} f(x) = f(a)$.

If f is continuous at every point in I we say that f is continuous on I.

Attention

The point a has to be a point in the domain of the function!

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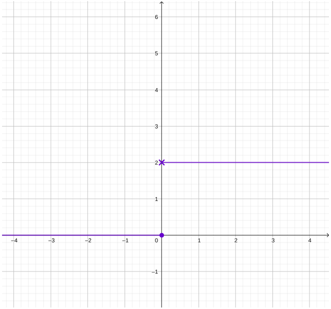
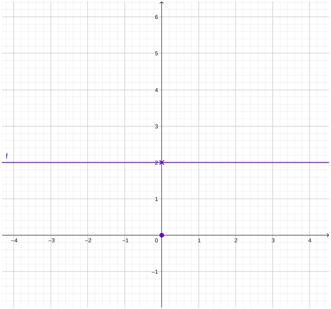
The point a has to be a point in the domain of the function!

Heuristically this means that you can draw the graph around a without lifting your pencil.

What can go wrong

- The limit does not exist.
- The function is not defined in the point. The function $f(x) = \frac{e^x - 1}{x}$ is not continuous at 0.
- The function is defined in the point, but its value is not equal f at nearby points.





Example

Exam question - First Learning outcome

Let a be a real number and let $f(x)$ the following function depending on a

$$f(x) = \begin{cases} \sqrt{ax} & \text{if } x > 2 \\ x - a & \text{if } x \leq 2 \end{cases}$$

Determine for which value of a this is a continuous function.

Properties of continuous functions

Let f and g be two continuous functions on their domain, and let $c \in \mathbb{R}$. Then we have that

- $c \cdot f(x)$, $f(x) + g(x)$, $f(x) - g(x)$, and $f(x)g(x)$ are continuous functions.
- $f(x)/g(x)$ is continuous at every point a such that $g(a) \neq 0$.
- The composition $f \circ g(x)$ is continuous whenever it is defined
- The inverse function $f^{-1}(x)$ is continuous.

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Attention!!!

A function can be continuous at a point, but its derivative can be discontinuous there. Think about $|x|$!

Important

Suppose f and g are two functions. Let a be in the domain of g and suppose that f is a continuous around $g(a)$ then we have that

$$\lim_{x \to a} (f \circ g)(x) = f\left(\lim_{x \to a} g(x)\right)$$

Section 2
More on limits

Infinite limits - Example

Consider the function

$$f(x) = \frac{1}{|x|}$$

What happens if we get closer and closer to 0?

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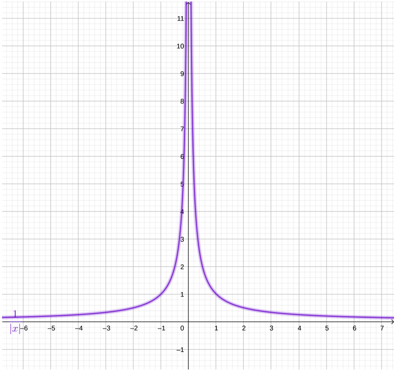

x	$f(x)$
-1	1
-0.1	10
-0.01	100
-0.001	1000
0	?
0.001	1000
0.01	100
0.1	10
1	1

Infinite limits

If, when we get closer and closer to a point a, $f(x)$ gets bigger and bigger (respectively smaller and smaller) then we say that when x tends to a, $f(x)$ tends to $+\infty$ (respectively $-\infty$). We write

$$\lim_{x \to a} f(x) = +\infty \quad \left(\lim_{x \to a} f(x) = -\infty \right)$$

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In this case, we say that the line $x = a$ is a vertical asymptote to the graph of the function. In general the same rule of calculations apply to limits at infinity as we can easily imagine what $a \pm \infty$, $a \times (\pm\infty)$, $\infty \times (\pm\infty)$ is. But we have to be careful.

Indetermined forms

$$0 \cdot (\pm\infty) \quad \frac{\pm\infty}{\pm\infty}, \quad +\infty - \infty, \quad \frac{0}{0}$$

If we get one of these we have to refine our calculations.

Left and right limits - Example

Suppose you throw a tetrahedral dice. Let $f(x)$ denotes the probability that the results is less than, or equal to x. How does $f(x)$ looks like?

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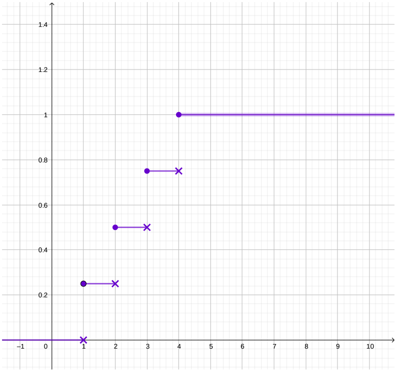

If we approach 1, 2, 3, and 4 on one side, we have that the limit exists. The problem is that it is different depending which side we are approaching from

Left and right limits

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Definition

If $f(x)$ tends to a number B when x approaches to a from smaller values (resp. bigger values) we say that the *B is the limit of $f(x)$ as x tends to a from below (resp. above)* and we write

$$\lim_{x\to a^-} f(x) = B \quad \left(\lim_{x\to a^+} f(x) = B\right)$$

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- The same rules we learned for limits work in this case
- We have that $\lim_{x\to a} f(x)$ exist iff both the limit from above and below exist and they are equal and equal to $f(a)$.
- One can speak of one sided continuity. One side continuous functions appear everywhere in probability/time series analysis.

Questions?

Limits at infinity

We can use the language of limits to describe the behaviour of a function when the argument x gets arbitrarily large or small.

Definition

We say that a function $f(x)$ tends to $L \in \mathbb{R} \cup \{\pm\infty\}$ when x goes to $+\infty$ (respectively $-\infty$) if we can make $f(x)$ arbitrarily close to L by sending x to infinity (- infintiy). We write

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On the exam (even under disguise)

Limits at infinity appears on exams. Usually in the "study of function" exercise. But they might also be under disguise in the integral exercise (we are learning integrals in Lecture 8). In order to compute definite integral on infinite intervale one needs to use limits at infinity!

Some useful limits to know

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$f(x)$	Limit to	Value/Condition
x^n with $n \in \mathbb{N}$	$+\infty$	$+\infty$
x^n with $n \in \mathbb{N}$	$-\infty$	$+\infty$ if n is even $-\infty$ if n is odd.
x^q with $q \in \mathbb{Q}$	$+\infty$	$+\infty$
x^q with $q \in \mathbb{Q}$	0^+	0 if $q \geq 0$, $+\infty$ if $q < 0$.
a^x	$+\infty$	$+\infty$ if $a > 1$, 1 if $a = 1$, 0, if $a < 1$.
a^x	$-\infty$	0 if $a > 1$, 1 if $a = 1$, $+\infty$, if $a < 1$.
e^x	$+\infty$	$+\infty$
e^x	$-\infty$	0
$\log_a(x)$	$+\infty$	$+\infty$ if $a > 1$, $-\infty$ if $a < 1$
$\log_a(x)$	0^+	$-\infty$ if $a > 1$, $+\infty$ if $a < 1$.
$\ln(x)$	$+\infty$	$+\infty$
$\ln(x)$	0^+	$-\infty$

Examples

Compute the following limits

- $\lim_{x \to +\infty} \frac{x^5 + x^4 + 3x + 1}{2x^5 + 2x + 1}$
- $\lim_{x \to +\infty} \ln(x+3) - \ln(x)$

Active Learning 1: Piecewise Pricing Continuity

Think-Pair-Share: Bulk Commodity Pricing

A Swedish logistics provider charges a firm total cost $C(x)$ based on the volume x (in tons) of freight transported:

$$C(x) = \begin{cases} 40x & \text{if } 0 \leq x \leq 100 \\ 35x + K & \text{if } x > 100 \end{cases}$$

Your Task:

1. **Think (1 min):** Using our limits from above and below rules (Slide 21), calculate $\lim_{x \to 100^-} C(x)$ and $\lim_{x \to 100^+} C(x)$.
2. **Pair (2 mins):** Find the exact setup fee value K that guarantees $C(x)$ is continuous at $x = 100$ so clients don't face sudden price shocks.
3. **Share (1 min):** Why is vertical alignment crucial for cost schedules?

Questions?

Section 3
L' Hôpital's rule

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Suppose that f and g are two functions such that

- they are differentiable around a, with the possible exception of a;
- $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

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- It works also with $a = \pm\infty$
- It works also with $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$.

Consequences

- $\lim_{x \to +\infty} \frac{x^n}{e^x} = 0$
- $\lim_{x \to +\infty} \frac{x^n}{\ln x} = +\infty$

Diagnosing l'Hôpital Traps

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Peer Instruction: Testing the Limit Premise

Consider the core target limit introduced on Slide 2:

$$\lim_{x \to 1} \frac{\ln(x) - x + 1}{x - 1}$$

Concept Check: Does this function satisfy the premise for l'Hôpital's rule?

A) No, because the denominator goes to 0 but the numerator goes to ∞.

B) No, because $\ln(1) = 1$.

C) Yes, it yields a valid $\frac{0}{0}$ form.

D) Yes, it yields an $\frac{\infty}{\infty}$ form.

Action: On my signal, hold up fingers for A, B, C, or D. Let's compute the derivative paths live on the board to confirm our outcome target!

Examples

Compute the following limits.

- (From an old exam) $\lim_{x\to 0+} \ln(x)x$
- $\displaystyle\lim_{x\to 0} \frac{e^{x^2}-1}{x^2}$
- $\displaystyle\lim_{x\to +\infty} \frac{(2x-1)^4}{(x-1)(x+3)}$
- $\displaystyle\lim_{x\to +\infty} \sqrt{\frac{x^2-e^{-x}+5}{x^2+2x+3}}$
- $\displaystyle\lim_{x\to +\infty} \frac{e^{2x}+x^5-\ln(x)}{\sqrt{1+e^{4x}}}$

Questions?

Section 4
Indefinite Integrals

Indefinite Integrals

Given a function f we want to find all functions F such that

$$F'(x) = f(x)$$

for all x. (F is called a primitive of f)

If F is a primitive of f, all the primitives of f are of the form $F(x) + C$, with $C \in \mathbb{R}$. The set of all primitives of F is denoted by

$$\int f(x)\,dx \qquad \text{(indefinite integral)}$$

- $\displaystyle\int e^x\,dx = e^x + C, \quad$ where $C \in \mathbb{R}$.
- $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C, \quad$ where $C \in \mathbb{R}$.
- $\displaystyle\int x^s\,dx = \frac{x^{s+1}}{s+1} + C \quad$ for $s \neq -1$, where $C \in \mathbb{R}$.

Given two functions f, g and $a \in \mathbb{R}$

- $\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$
- $\int af(x)\, dx = a \int f(x)\, dx$

- $\int (3\sqrt{x} + 6x^2 - x^{-1} + e^x)\, dx =$

Active Learning 3: Term-by-Term Primitive Workshop

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Group Challenge: Mastering the Indefinite Baseline (5 Minutes)

Let us evaluate our target integral from Slide 38 together:

$$\int \left(3\sqrt{x} + 6x^2 - x^{-1} + e^x\right) dx$$

Work with your team to integrate each component cleanly:

1. **Term 1:** Write $3\sqrt{x}$ as $3x^{1/2}$. What power rule coefficient scales it?
2. **Term 2 & 3:** What happens to the primitive of x^{-1} ($\frac{1}{x}$)? Do we need absolute value bars?
3. **Step 3:** Combine all terms and state what essential constant symbol must be added to complete an indefinite expression.

Thank you for your attention!