MM3001

Lecture 8: Optimization and graphing

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Review & Clarifications:

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Opening the Floor for Clarifications

Before we dive into optimization algorithms today, what questions do you have regarding last lecture's material on **Limits, Continuity, or Indefinite Integrals**?

60-Second Flash Recall: Warming Up Your Mathematical Brain

To help recall core mechanics while you look through your notes for questions:

- If you use l'Hôpital's rule on a limit expression, what is the mandatory requirement for the numerator and denominator parameters?
- What is the conceptual difference between a continuous function and a differentiable function?

Take a moment to check your notes from last week and ask any

Lecture Goal and Outcome

Goal: Understand how the graph of a function looks like by looking at its law. Solve optimization problems.

Learning Outcome: At the end of today lecture you will be able to solve problem like the following

Problem

A firm production function is $Q(L) = 12L^2 - \frac{1}{20}L^3$, where $L \in [0, 200]$ denotes the number of workers.

1. What number of worker maximise Q?
2. Let L^* denote the size of the workforce that maximizes the output per worker $Q(L)/L$. Find L^*.

Why You Should Care: The Cost of Missing the Peak

The Economist's Dilemma

Imagine you operate an airline. If your price ticket is too cheap, your planes fly full but you make little revenue. If your price ticket is too high, your seats stay empty.

The Mathematical Reality of Markets:

- Revenue curves are concave bowls (parabolas). At the very peak, the curve goes flat for a microscopic instant.
- A flat curve means the slope (derivative) is exactly **zero** ($f'(x) = 0$).
- By solving $f'(x) = 0$, you locate the perfect equilibrium price without relying on blind guesswork. Today we master this exact mapping mechanism!

Lecture Plan

- Extreme points (definition and existence) (9.1, 9.4)
- The hunt for critical points (9.2)
- Local extreme points (9.6)
- The second derivative test - Inflection points (8.6)

Section 1
Extreme points

Extreme points

Definition

Let f be a function with domain D, we say that

1. $a \in D$ is a (global) maximum point for f and $f(a)$ is the maximum value of f if
$$f(x) \leq f(a) \text{ for every } x \in D$$
2. $a \in D$ is a (global) minimum point for f and $f(a)$ is the minimum value of f if
$$f(x) \geq f(a) \text{ for every } x \in D$$

Example

The point $x = 0$ is a minimum point of $x^2 - 30$. The minimum value of the function $f(x) = x^2 - 30$ is $f(0) = -30$.

They exists!

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Extreme value Theorem

A continuous function defined on a closed interval $[a, b]$ has (at least) a maximum and a minimum point.

Example

Find the maximum and the minimum points of $f(x) = x^2 + 5$ on $[-2, 2]$. Give the maximum and the minimum value of f on $[-2, 2]$.

The Endpoint Boundary Trap

Peer Instruction: Testing Global Boundaries (3 Minutes)

Consider an utility function defined on a closed interval $x \in [1, 5]$:

$$f(x) = x^2 - 4x + 10$$

Concept Check: What are the absolute maximum and minimum coordinates?

A) Min at critical point $x = 2$; Max doesn't exist.

B) Min at endpoint $x = 1$; Max at $x = 5$.

C) Min at critical point $x = 2$; Max at endpoint $x = 5$.

D) Min at $x = 2$; Max at $x = 1$.

Action: On my count of three, raise fingers for A, B, C, or D. Let's remember the Extreme Value Theorem on Slide 7: We must always compare critical values to boundary points!

Questions?

Section 2
The hunt for extreme points

Critical points and extreme points

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Definition

A point c where a function f is differentiable and such that $f'(c) = 0$ is called critical point for f.

Necessary first order condition

Suppose that the function f, defined on $[a, b]$ is differentiable in (a, b). If $c \in (a, b)$ is a maximum or minimum point for f on $[a, b]$, then c is a critical point for f, that is

$$f'(c) = 0$$

Consequences

This tell us that min/max point on a closed interval $[a, b]$ can be

- on critical points
- on the extremes a and b
- in points where f is not differentiable

You get a finite number of candidates, compute f at those point, compare and decide!

Problem

A firm production function is $Q(L) = 12L^2 - \frac{1}{20}L^3$, where $L \in [0, 200]$ denotes the number of workers.

1. What number of worker maximise Q?
2. Let L^* denote the size of the workforce that maximizes the output per worker $Q(L)/L$. Find L^*.

Active Learning 2: Solving the Output Per Worker Goal

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Think-Pair-Share: Maximizing Average Productivity (5 Minutes)

Let us solve Part 2 of today's initial lecture problem (Slide 2 & 11). The firm's production layout is $Q(L) = 12L^2 - \frac{1}{20}L^3$ on $L \in [0, 200]$.

Your Task:

1. **Think (2 mins):** Write the function representing output per worker, $A(L) = \frac{Q(L)}{L}$. Then compute its derivative $A'(L)$.
2. **Pair (2 mins):** Set $A'(L) = 0$ to find the critical worker threshold L^* that maximizes efficiency.
3. **Share (1 min):** Let's verify why maximizing output per worker occurs at a different scale than maximizing total output $Q(L)$.

Questions?

Section 3
Local extreme points

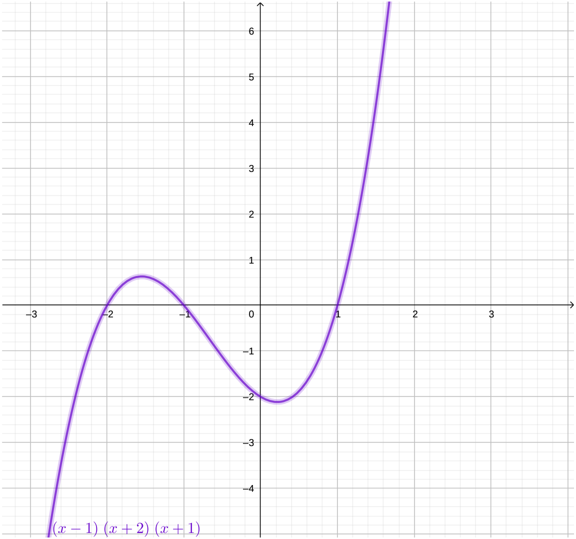
Local extreme points

Definition

Let f be a function with domain D, we say that

1. $a \in D$ is a local maximum point for f if there is an interval $I = (\alpha, \beta)$ containing a such that
$$f(x) \leq f(a) \text{ for every } x \in I \cap D$$
2. $a \in D$ is a local minimum point for f if there is an interval $I = (\alpha, \beta)$ containing a such that
$$f(x) \geq f(a) \text{ for every } x \in I \cap D$$

Example



The first derivative test

Necessary condition

Let f be defined on an interval I and c be an element of the interior of I such that f is differentiable at c. If f has a local extreme point at c, then c is a critical point for f.

First derivative test

Consider the function $f(x)$ and let c be a critical point for f

- If $f'(x)$ change sign at c from positive to negative, then $x = c$ is a local maximum point.
- If $f'(x)$ change sign at c from negative to positive, then $x = c$ is a local minimum point.
- If $f'(x)$ does not change sign then c is not a local extreme (it could be an inflection point)

Example from an old exam

Consider the function

$$f(x) = \sqrt{x}e^{-x}$$

1. Find the domain of f
2. Find in which interval the function is increasing/decreasing. Find all local extreme point and decide their type.
3. Compute $\lim_{x\to\infty} f(x)$
4. Sketch the graph

Example from an old exam II

Consider the function

$$f(x) = \frac{(4+x)^2}{x-1}$$

1. Find the domain of f
2. Find in which intervals the function is increasing/decreasing. Find all local extreme points and decide their type.
3. Compute $\lim_{x\to\infty} f(x)/2x$
4. Has f and global maximum or minimum?

Example from an old exam III

Consider the function

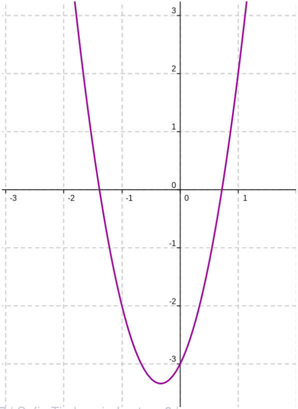
$$f(x) = x^3(x-1)$$

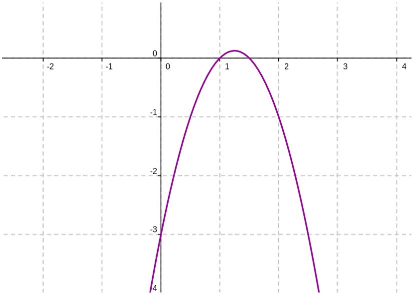
1. Find in which intervals the function is increasing/decreasing.
2. Find the maximum and minimum value of f in the interval $[-1, 1]$.
3. Compute $\lim_{x\to\infty} f(x)/e^x$,

Section 4
Convexity and second derivatives

Convexity

We are not going to give the formal definition of convex and concave functions. We will just say, really informally, that a function is convex around a point *a* if its graph looks like a bowl. It is concave if it looks like a cap.





The second derivative tests

Test for convexity

Let f be a function twice differentiable in an interval I.

1. If $f''(c) \geq 0$ for every point c in I we have that f is convex on I.
2. If $f''(c) \leq 0$ for every point c in I we have that f is concave on I.

Test for local extrema

Let f be a function twice differentiable in an interval I, and let c be a critical point of f, lying in the interior of I.

1. If $f''(c) > 0$ we have that c is a local minimum.
2. If $f''(c) < 0$ we have that c is a local maximum
3. If $f''(c) = 0$ it can be neither or either.

Inflection points

Inflection points

If f changes from being convex to concave or from being concave to convex in c, then c is an inflection point.

In inflection points, we have $f''(c) = 0$, but not all points for which $f''(c) = 0$ are inflection points.

Example (Old Exam)

Consider the function

$$f(x) = (x^2 - 9)e^{5x}$$

1. Find the the places where the function is 0, all local extreme points and decide their type. Find where the function is concave and convex.
2. Sketch the graph and compute, the limits of $f(x)$ when x tends to $+\infty$ and $-\infty$, if they exists.

Active Learning 3: Inflection Point Diagnostic Check

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Group Challenge: Tracking Corporate Growth Inflections (4 Minutes)

Suppose a firm's long-run revenue trajectory follows a smooth curve. The second derivative test gives us structural optimization details (Slide 22).

Analyze this scenario with your team:

- If $f''(c) = 0$, does that automatically prove that c is an inflection point? Look closely at the theorem conditions !
- Can a point be a local maximum if the second derivative is positive ($f''(c) > 0$)? Explain using the visual bowl analogy.

Questions?

Thank you for your attention!