MM3001

Lecture 9: Integration

Sofia Tirabassi

Review & Clarifications

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Opening the Floor for Clarifications

Before we dive into advanced integration methods, what questions do you have regarding last lecture's material on **Optimization, Convexity, or Curve Sketching**?

60-Second Flash Recall: Warming Up Your Mathematical Brain

To help recall core mechanics while you look through your notes for questions:

- If an economic function satisfies $f'(c) = 0$ and $f''(c) < 0$, what does point c represent?
- Can a local maximum occur at a boundary endpoint where the derivative is not zero?

Take a moment to check your notes from last week and ask any clarifications!

The Indefinite Integral & Primitives

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The Inverse Process of Differentiation

An **indefinite integral** finds the family of all functions whose derivative equals the integrand. We call these parent functions **primitives**.

$$\int f(x)\,dx = F(x) + C \quad \Longleftrightarrow \quad F'(x) = f(x)$$

Essential Baseline Rules for Economists:

- **Power Rule ($n \neq -1$):** $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$
- **The Logarithmic Case:** $\int \frac{1}{x}\,dx = \ln|x| + C$
- **The Exponential Case:** $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$

Why do we need a constant $+C$?

Because the derivative of any constant is 0. Differentiating $F(x) + 5$ or $F(x) - 100$ yields the exact same $f(x)$.

Lecture Goal and Outcome

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Goals:

- Introduce the notion of integration, properties and methods.

Learning Outcome: At the end of the lecture you will be able to:

- Integrate by substitution and by parts;
- Calculate the signed area under a curve;
- Calculate improper integrals.

Why You Should Care

Real-World Economic Pillars of Integration:

- **Accumulating Total Value:** If you know the continuous rate of a country's investment flow or a firm's profit generation over time, integrating that rate over a time horizon $[0, T]$ gives you the total accumulated wealth.
- **Market Surplus & Welfare:** The area under a demand curve down to the equilibrium price represents the **Consumer Surplus**. Integration allows us to calculate this area precisely, even when demand curves are complex and non-linear.
- **Lifetime Present Value:** Discounting future cash streams using continuous compounding requires integrating continuous flows over infinite intervals:

$$\text{Total PV} = \int_0^\infty f(t)e^{-rt}\,dt$$

Lecture Plan

- §10.2 Area and Definite Integrals
- §10.3 Properties of Definite Integrals
- §10.5 Integration by Parts
- §10.6 Integration by Substitution
- §10.7 Infinite Intervals of Integration

Section 1
Integration methods

Recall two rules we have seen in the course:

- Product-rule for derivatives

$$\frac{\mathrm{d}(u(x)v(x))}{\mathrm{d}x} = \frac{\mathrm{d}u(x)}{\mathrm{d}x}v(x) + u(x)\frac{\mathrm{d}v(x)}{\mathrm{d}x},$$

- $$\int \frac{\mathrm{d}w(x)}{\mathrm{d}x}\mathrm{d}x = w(x) + C$$

Combining these two together we have

$$\begin{aligned} u(x)v(x) + C &= \int \frac{\mathrm{d}(u(x)v(x))}{\mathrm{d}x} \\ &= \int \frac{\mathrm{d}u(x)}{\mathrm{d}x}v(x)\mathrm{d}x + \int u(x)\frac{\mathrm{d}v(x)}{\mathrm{d}x}\mathrm{d}x, \end{aligned}$$

Integration by parts formula

$$\int u(x)\frac{\mathrm{d}v(x)}{\mathrm{d}x}\mathrm{d}x = u(x)v(x) - \int \frac{\mathrm{d}u(x)}{\mathrm{d}x}v(x)\mathrm{d}x$$

Examples

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Integration by parts formula

$$\int u(x)v'(x)\mathrm{d}x = u(x)v(x) - \int u'(x)v(x)\mathrm{d}x$$

- $\int xe^x \mathrm{d}x$

- $\int \ln x \mathrm{d}x$

Integration by substitution

$$\int f(y(x))y'(x)\mathrm{d}x = \int f(y)\mathrm{d}y$$

Say that $F(y)$ is a primitive for f, and that $y = y(x)$. The chain rule gives

$$\frac{\mathrm{d}F(y(x))}{\mathrm{d}x} = \frac{\mathrm{d}F(y)}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = f(y(x))y'(x)$$

Integrating gives

$$\int f(y(x))y'(x)\mathrm{d}x = F(y(x)) + C = \int f(y)\mathrm{d}y$$

Example 1

- Find a change of variables such that $f(x)$ is close to the form $g(s(x))s'(x)$.
- Write $\int f(x)\mathrm{d}x$ in the form $\int g(s)\mathrm{d}s = G(s) + C$;
- Write $G(s)$ as $G(s(x))$.

Calculate

1. $$\int (9 - x^3)x^2 \mathrm{d}x$$
2. $$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \mathrm{d}x$$
3. $$\int \frac{t \ln(t^2 + 1)}{t^2 + 1} \mathrm{d}t.$$
4. $$\int \ln(1 + 2t) \left(\frac{1}{2} + t\right)^2 \mathrm{d}t$$

Choosing Your Integration Tool

Peer Instruction: Strategy Check (3 Minutes)

Look at the two integrals below. We need to decide whether to attack them using **Integration by Parts** or **Substitution**.

Expression 1: $\int x^2 e^{x^3}\, dx$ Expression 2: $\int x^2 e^x\, dx$

Concept Check: Which strategy matches which expression?

A) Both require substitution.

B) Both require parts.

C) 1 requires parts; 2 requires substitution.

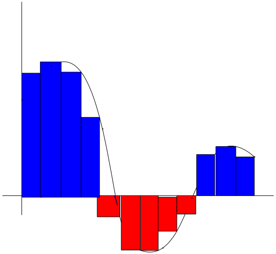
D) 1 requires substitution; 2 requires parts.

Action: On my signal, hold up fingers for A, B, C, or D. Let's look for nested derivatives (x^2 is the derivative of the inner exponent x^3) to spot substitutions instantly!

Section 2
Definite Integral

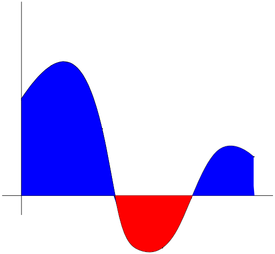
Definite integral

Playing in the casino, we record every hour how much money we gain/loose at the end of that hour. Say we denote the profit (which can be either positive or negative) at time t as $P(t)$. Say that we plot that as below:



What is the net earning after 12 hours?
A: The area of the blue rectangles minus the are of the red rectangles.

What about if we record it every second? Or millisecond?
A: The area of above the x-axis and under the graph minus area under the x-axis and above the graph.



Let $f : [a, b] \to \mathbb{R}$. We define the signed area under the graph of f the difference between the area over the x-axis, minus that under the x-axis. We denote that area as

$$\int_a^b f(x)\mathrm{d}x := A_+ - A_-.$$

How can we calculate $\int_a^b f(s)\mathrm{d}s$

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Fundamental Theorem of Calculus

Let $f : [a, b] \to \mathbb{R}$ be continuous. Define, for all $x \in [a, b]$ the area integral $A(x) := \displaystyle\int_a^x f(s)\mathrm{d}s$,

1. $A(x)$ is a primitive for f, i.e. $A'(x) = f(x)$.
2. If $F(x)$ is any primitive for f

$$\int_a^b f(s)\mathrm{d}s = F(b) - F(a) = [F(x)]_a^b\,,$$

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Calculate the following integrals

1. (2019) $\displaystyle\int_6^{49} \frac{1}{2x-5}\mathrm{d}x$
2. (2017) $\displaystyle\int_1^{e^2} x^3 \ln x \mathrm{d}x$

Section 3
Improper Integrals

Improper integral

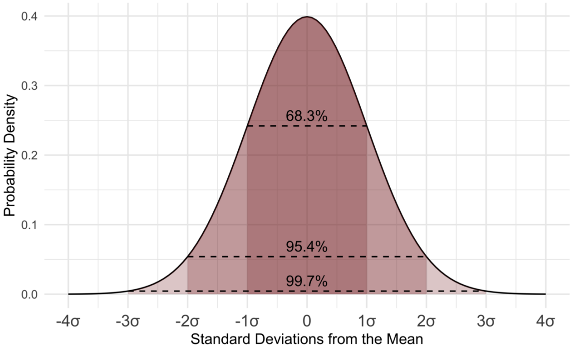
1. Let f be a continuous function on $[a, \infty)$. Define
$$\int_a^\infty f(x)\,dx \quad \text{to be} \quad \lim_{b\to\infty}\int_a^b f(x)\,dx,$$
2. Let f be a continuous function on $(-\infty, b]$. Define
$$\int_{-\infty}^b f(x)\,dx \quad \text{to be} \quad \lim_{a\to-\infty}\int_a^b f(x)\,dx,$$
3. Let f be a continuous function on $(-\infty, \infty)$. Let c be any real number; define
$$\int_{-\infty}^\infty f(x)\,dx \quad \text{to be} \quad \lim_{a\to-\infty}\int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx,$$

An improper integral is said to **converge** if its corresponding limit exists; otherwise, it **diverges**.

Motivational example

If $X \sim N(\mu, \sigma)$ then its distribution function is given by

$$P(X \leq t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} \mathrm{d}x,$$



Motivational example II

Suppose that an income in a population is distributed according to the Pareto distribution

$$P(X \geq x) = Cx^{\alpha}$$

The expected value for the income is

$$\int_0^{+\infty} -C\alpha x^{\alpha},$$

Bounding the Infinite Horizon

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Think-Pair-Share: The $1/x^p$ Paradox (4 Minutes)

Look at the first two improper integral examples listed on Slide 18:

Case 1: $\int_1^\infty \frac{1}{x^2}\,dx$ Case 2: $\int_1^\infty \frac{1}{x}\,dx$

Your Task:

1. **Think (1 min):** Both denominators grow larger as $x \to \infty$, meaning both curves shrink toward 0. Does this guarantee they both enclose a finite area?
2. **Pair (2 mins):** Apply the limit rules from Slide 15. Solve both cases with your bänkkamrat.
3. **Share (1 min):** Why does Case 1 converge to a clean integer while Case 2 blows up to infinity?

Examples

1. $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x$
2. $\displaystyle\int_1^{\infty} \frac{1}{x}\,\mathrm{d}x$
3. $\displaystyle\int_0^{+\infty} e^{-x}\,\mathrm{d}x$

Thank you for your attention!