MM3001

Lecture 10: Functions of several variables

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Review: Retrieval Practice

The Marginal to Total Challenge

An economic analyst determines that the marginal cost (MC) of a clean-tech startup's production is modeled by:

$$MC(q) = 6q^2 - 10q + 50$$

The firm's fixed setup costs are known to be $F = \$500$.

Turn to your neighbor and solve:

1. Reconstruct the **Total Cost function**, $C(q)$, using integration: $C(q) = \int MC(q)\, dq$.
2. How did you use the fixed cost F to solve for the constant of integration (C)?
3. Calculate the total cost of producing exactly $q = 10$ units.

Lecture Goal and Outcome

Goals:

- Identify functions of two variables
- Determine the first-order partial derivatives of functions
- Determine second-order partial derivatives,
- Use the chain rule

Why You Should Care: Breaking the Single-Variable Illusion

The Old Way (Introductory Math)

- Assume a firm only sells *one* product.
- Costs depend only on one output level: $C(x)$.
- Unrealistic for modern business.

The Real Way (Economic Reality)

- Firms produce product portfolios (A and B).
- Interactions matter! (Shared factories, cross-product assembly lines).

The Core Multi-Variable Challenge

Daily multi-good cost: $C(x, y) = 4x^2 + xy + y^2 + 400x + 200y + 500$
Market selling prices: $P_A = 15$ and $P_B = 9$

1. How do we find the unique mix of (x, y) that maximizes profit?
2. What if supply chains tie them together over time, so $x = x(t)$ and $y = y(t)$?

Why should you care

Have you ever heard of the GREEKS?

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$$\Delta, \quad V, \quad \Theta$$

Are first order partial derivatives of the value of a portfolio with respect to

- Δ the price of a commodity
- V the variance of the price of a commodity (usually you have to differentiate implicitly to find it)
- Θ the time.

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- Δ the price of a commodity
- V the variance of the price of a commodity (usually you have to differentiate implicitly to find it)
- Θ the time.

$$\Gamma$$

is the second order partial derivative of the value of a portfolio with respect of the price of a commodity (twice)

Lecture Plan

- §14.1 Functions of Two Variables
- §14.2 Partial Derivatives with Two Variables
- §14.3 Geometric Representation
- §14.4 Surfaces and Distance
- §15.1 A Simple Chain Rule

Section 1
Functions of several variables

Functions of two variables

A company produces two goods (A, B) in quantities x, y respectively. The total cost of producing them is given by

$$C(x, y) = 1000 + 20x + 40y,$$

Let $D \subset \mathbb{R}^2$. A function $f : D \to \mathbb{R}$ is a rule, that for each $(x, y) \in D$ it gives us a value $f(x, y)$. D is called the domain of definition of f.

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Examples:

1. $f(x, y) = x^2 + y^2$
2. $f(x, y) = x^2 - y^2$
3. $f(x, y) = x^2 e^{x+y}$
4. $f(x, y) = 1/(x^2 + 2y^2 + 1)$

The domain of definition of these functions is ...
If $D \subset \mathbb{R}^n$, we say that f is a function of n-variables.

Domain Mapping & Error Catching (7 Mins)

The Multi-Asset Utility Constraint

An investor has a risk-return function defined by:

$$f(x, y) = \ln(x - y) + \sqrt{1 - x + y}$$

Team Strategy Task:

1. **Analyze Component 1:** For $\ln(x - y)$ to exist, what inequality must hold? Is the boundary included or excluded?
2. **Analyze Component 2:** For $\sqrt{1 - x + y}$ to yield real values, what must hold?
3. **Synthesize:** Sketch or describe the overlapping target zone where an economist can safely evaluate this function.

Hint: Watch your strict inequality ($>$) versus weak inequality ($\geq$) boundaries carefully!

Partial derivatives

We define the partial derivative of f with respect to x at (x_0, y_0) as

$$\frac{\partial f}{\partial x}(x_0, y_0) = f'_x(x_0, y_0) := \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

We fix y_0, and differentiate the function as a one-variable function in x

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Similarly we define the partial derivative of f with respect to y at (x_0, y_0) as

$$\frac{\partial f}{\partial y}(x_0, y_0) = f'_y(x_0, y_0) := \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

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Example: If $f(x, y) = x^2 + xy + 3y^2$ then

$$f'_x(x, y) = 2x + y, \qquad f'_y(x, y) = x + 6y$$

Examples

We can use all the rules we have learnt to compute partial derivatives:
If $f(x, y) = (x^2 + y)e^{xy}$ then

$$\begin{aligned} f'_x(x, y) &= \left[\frac{\partial}{\partial x}(x^2 + y)\right] e^{xy} + (x^2 + y)\left[\frac{\partial}{\partial x} e^{xy}\right] \\ &= [2x]\, e^{xy} + (x^2 + y)\,[ye^{xy}] \\ &= (2x + x^2 y + y^2)e^{xy} \end{aligned}$$

Similarly

$$f'_y(x, y) =$$

Higher order derivatives

We can define

$$f''_{xx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y) \qquad f''_{yx}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y)$$

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Example If $f(x,y) = x^3 e^{y^2}$ then $f'_x(x,y) = 3x^2 e^{y^2}$, $f'_y(x,y) = 2yx^2 e^{y^2}$,

$$f''_{xx}(x,y) = 6xe^{y^2} \qquad f''_{yx}(x,y) = 6x^2 ye^{y^2}$$

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If the second partial derivatives exists and are continuous then
$f''_{yx}(x,y) = f''_{xy}(x,y)$

Geometric representation

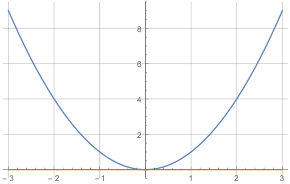
A real number x — A point on the line $\mathbb{R}$

A pair of numbers (x, y) — A point on the plane $\mathbb{R}^2$

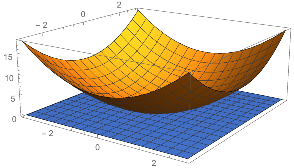
A triple of number (x, y, z) — A point in space $\mathbb{R}^3$

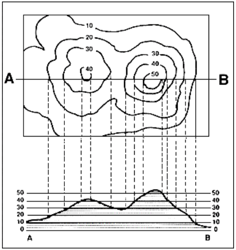
The graph of $f : D \to \mathbb{R}$ is the set of points

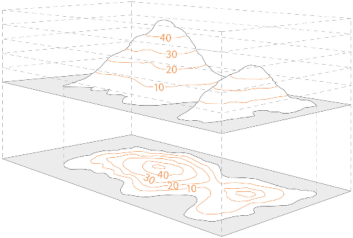
If $D \subset \mathbb{R}$
in $\mathbb{R}^2$ of the form $(x, f(x))$
with $x \in D$.

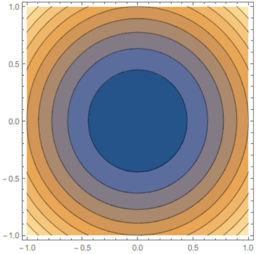


If $D \subset \mathbb{R}^2$
in $\mathbb{R}^3$ of the form $(x, y, f(x, y))$ with $(x, y) \in D$.









A way of representing graphically a function f is by plotting in the (x, y)-plane the sets of points satisfying that

$$f(x, y) = c$$

for a range of constants c. These are called level curves of f.

Distance

Given two points in the plane $X = (x, y)$, $Y = (a, b)$ the distance between them is

$$d(X, Y) := \sqrt{|x - a|^2 + |y - b|^2},$$

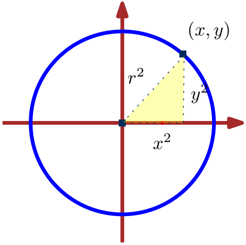
So, if $Y = (0, 0)$, $f(x, y) := \sqrt{x^2 + y^2}$ represents the distance of (x, y) to the origin. The level curves of the form

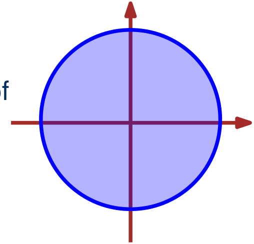
$$x^2 + y^2 = r^2, \quad r > 0$$

are circle of radius c. Also, the sets of points (x, y) such that

$$x^2 + y^2 \leq r^2, \quad r > 0$$

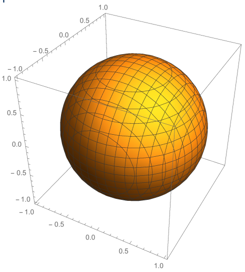
are disks of radius r.





Given two points in the plane $X = (x, y, z)$, $Y = (a, b, c)$ the distance between them is

$$d(X,Y) := \sqrt{|x-a|^2 + |y-b|^2 + |z-c|^2}$$



So, $\sqrt{x^2+y^2+z^2}$ represents the distance of (x, y, z) to the origin.
The set of points (x, y, z) such that

$$x^2 + y^2 + z^2 = r^2, \quad r > 0$$

is a sphere of radius r. Also, the set of points (x, y, z) such that

$$x^2 + y^2 + z^2 \leq r^2, \quad r > 0$$

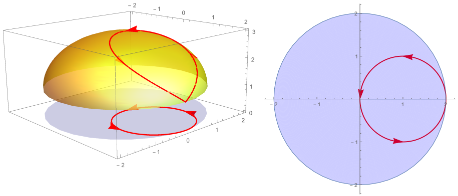
is a (solid) ball of radius r.

Section 2
The chain rule

Chain rule

Assume that $f(x, y)$ is a function of two variables and both $x = x(t)$, and $y = y(t)$ are functions of one variable t. Then we can consider the one-variable function

$$g(t) := f(x(t), y(t))$$



The points $t \mapsto (x(t), y(t))$ can be thought describing a curve on the domain of f. So g describes the values of f along that curve.

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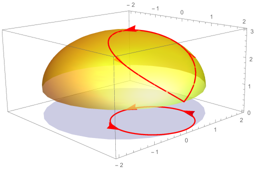

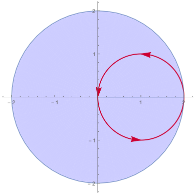

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The chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) = \frac{\partial}{\partial x} f(x(t), y(t)) \frac{\mathrm{d}}{\mathrm{d}t} x(t) + \frac{\partial}{\partial y} f(x(t), y(t)) \frac{\mathrm{d}}{\mathrm{d}t} y(t)$$

Peer Problem Solving - The Chain Rule (8 Mins)

The Scenario

Let a firm's production layout be $f(x, y) = x^2 y + 3y$. Both resource inputs depend dynamically on time t, where $x(t) = t^2$ and $y(t) = t^3$.

Apply the Chain Rule Formula:

$$\frac{dg}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

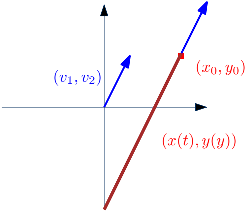
Work through these steps with your partner:

1. Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
2. Compute the time derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.
3. Substitute everything into the formula to find the total rate of change with respect to time, $\frac{dg}{dt}$.

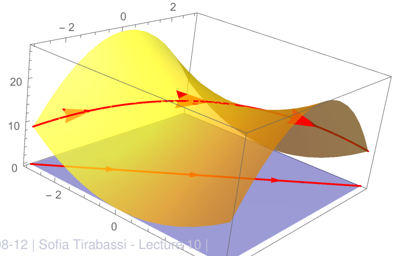
Example: Directional derivatives

Fix (v_1, v_2) a vector in the plane, and $(x_0, y_0) \in \mathbb{R}^2$ and let

$$\begin{cases} x(t) = x_0 + tv_1 \\ y(t) = y_0 + tv_2 \end{cases} \qquad t \in \mathbb{R}$$



This describe a line through the point (x_0, y_0) with direction (v_1, v_2).



So $f(x(t), y(t))$ gives the values of f along that line.

$$\begin{cases} x(t) = x_0 + tv_1 \\ y(t) = y_0 + tv_2 \end{cases}$$

$t \in \mathbb{R}$. By the chain rule

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f(x(t), y(t)) &= \frac{\partial}{\partial x} f(x(t), y(t)) v_1 + \frac{\partial}{\partial y} f(x(t), y(t)) v_2 \\ &= \begin{bmatrix} \frac{\partial}{\partial x} f(x(t), y(t)) & \frac{\partial}{\partial y} f(x(t), y(t)) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \end{aligned}$$

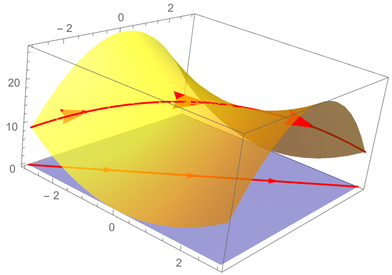
For $t = 0$

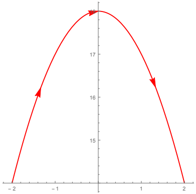
$$D_v f(x_0, y_0) := \frac{\mathrm{d}}{\mathrm{d}t} f(x(t), y(t))|_{t=0} = \overbrace{\begin{bmatrix} \frac{\partial}{\partial x} f(x_0, y_0) & \frac{\partial}{\partial y} f(x_0, y_0) \end{bmatrix}}^{Df(x_0, y_0)} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$D_v f(x_0, y_0)$ is called the directional derivative of f in the direction v.
$Df(x_0, y_0)$ is called the total derivative of f at (x_0, y_0).

Lets take in particular

$$f(x,y) = 18 + x^2 - 2y^2, \quad \begin{cases} x(t) = t \\ y(t) = t \end{cases}, t \in \mathbb{R} \quad g(t) = f(t,t) = 18 - t^2$$





Then $D_{(1,1)} f(0,0) = g'(0) = 0$.

Thank you for your attention!