MM3001

Lecture 12: Optimisation of functions of two variables

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Review & Clarifications:

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Opening the Floor for Clarifications

Before we dive into constraints and Lagrange Multipliers today, what questions do you have regarding last lecture's material on **Hessian Matrices and Local Extremum Classification**?

60-Second Flash Recall: Warming Up Your Mathematical Brain

To help recall core mechanics while you look through your notes for questions:

- If a stationary point yields $\det H > 0$ and $f''_{xx} < 0$, what landscape feature does this represent?
- What unique behavior defines a saddle point?

Take a moment to check your notes from last week and ask any clarifications!

Lecture Goal and Outcome

Goals:

- Use the second-order partial derivatives to classify the stationary points of a function of two variables.
- Solve constrained optimisation problems with the method of substituiton and Lagrange multipliers

Learning Outcome: At the end of the lecture you will be able to solve problems like the following:

Constrained Problem

A monopolistic producer of two goods, G_1 and G_2, has a joint cost function $C = 10Q_2 + Q_2Q_1 + 10Q_1$ where Q_1 and Q_2 denote the quantities of G_1 and G_2 respectively. If P_1 and P_2 denote the corresponding prices then the demand equations are
$P_1 = 50 - Q_2 + Q_1$, $P_2 = 30 + 2Q_2 - Q_1$.
Find the maximum profit if the firm is contracted to produce a total of at most 16 goods of either type.

Active Learning

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Think-Pair-Share: Setting up the Profit Objective (5 Minutes)

Let us set up our core lecture goal from Slide 2. A firm sells two goods with prices:

$$P_1 = 50 - Q_2 + Q_1 \qquad P_2 = 30 + 2Q_2 - Q_1$$

Their joint cost function is $C = 10Q_2 + Q_2Q_1 + 10Q_1$.

Your Task with Your Bänkkamrat:

1. **Think (2 mins):** Write down the Total Revenue equation $TR = P_1Q_1 + P_2Q_2$. Expand it completely.
2. **Pair (2 mins):** Subtract the joint cost C to state the full Profit objective function $\Pi(Q_1, Q_2) = TR - C$. Identify any terms that cancel out!
3. **Share (1 min):** If the firm must produce a total of at most 16 goods, write down the structural inequality constraint.

Lecture Plan

- Optimisation with constraints: Substitution and Lagrange multipliers.

Recall: Necessary condition

If a function $f : D \to \mathbb{R}$ has a local extrema at a point (x_0, y_0) that is an interior point of D then

$$f'_x(x_0, y_0) = 0 \quad \text{and} \quad f'_y(x_0, y_0) = 0$$

That is (x_0, y_0) is a stationary point.

Why You Should Care: Scarcity and the Budget Line

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The Economic Reality of Constraints

In the real world, optimization is rarely unconstrained. Consumers want to maximize utility but are limited by a **budget constraint**. Firms want to maximize output but are limited by **resource caps**.

Visualizing Bounded Realities:

- An unconstrained profit peak might require hiring 500 workers.
- But if your factory only fits 100 people, the unconstrained peak becomes completely irrelevant.
- Today we learn how to hunt along the physical boundaries of a market to find the best possible solution under pressure.

Section 1
Constrained Optimisation

The extreme value theorem

1D case

Let $f : [a, b] \to \mathbb{R}$ differentiable
$[a, b]$ is closed and bounded

1. There exists $x_m, x_M \in [a, b]$ such that for all $x \in [a, b]$

$$f(x_m) \leq f(x) \leq f(x_M)$$

2. The Maximum/minimun points are either
    1. A stationary point inside $[a, b]$; or
    2. A point in the border (either a or b).

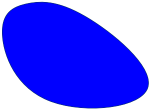
2D case

Let $f : D \to \mathbb{R}$ differentiable
D is closed and bounded

1. There exists $x_m, x_M \in D$ such that for all $x \in D$

$$f(x_m) \leq f(x) \leq f(x_M)$$

2. The Maximum/minimun points are either
    1. A stationary point inside D; or
    2. A point in the border (is a curve).



Optimisation problem

The strategy is essentially the same one as for functions in one variable.
Given $f : D \to \mathbb{R}$, where D is closed and bounded.

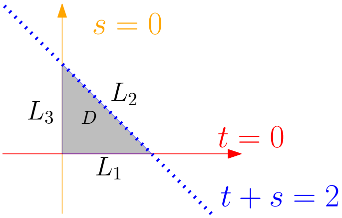
1. Make a list consisting of
   - Stationary points inside D;
   - Extremal "Boundary points"
2. Evaluate f at all the points in the list
3. Find the largest/smallest value

To find the extremes in the boundary we need to describe it/parametrise it.

Example

(2018) A quantity C is determined by inputs s, t, according to the formula $C(s,t) = st(s+t-1)$. Find the maximum and minimum possible values for C subjected to the constraints $s \geq 0$, $t \geq 0$ and $s + t \leq 2$.

Lets start by visualising the domain D:



The border of D consist of three pieces:

1. L_1 is the set of points (s, t) such that $t = 0$, $s \in [0, 2]$;
2. L_3 is the set of points (s, t) such that $s = 0$, $t \in [0, 2]$;
3. L_2 is the set of points (s, t) such that $t = 2 - s$, $s \in [0, 2]$.

Stationary points inside D

Recall that $C(s,t) = st(s+t-1)$
The partial derivative are:

$$\begin{cases} C'_s(s,t) = t(2s+t-1) = 0 \Leftrightarrow t = 0 \text{ or } 2s+t-1=0 \\ C'_t(s,t) = s(2t+s-1) = 0 \Leftrightarrow s = 0 \text{ or } 2t+s-1=0 \end{cases}$$

So the stationary points are those (s,t) satisfying:

1. $t = 0 = s$ i.e. $(0,0)$.
2. $t = 0$ and $2t+s-1=0$ i.e. $(1,0)$;
3. $s = 0$ and $2s+t-1=0$ i.e. $(0,1)$;
4. $2s+t-1=0$ and $2t+s-1=0$ i.e. $(1/3,1/3)$;

Only $(s,t) = (1/3,1/3)$ lies inside D.

On the Boundary

Recall that $C(s,t) = st(s+t-1)$
We look for the maximum/minimum values of the function when it is restricted to the border (i.e. $C(s,t)$ when (s,t) belong to the border):

- Alongside L_1 ($t = 0$, $s \in [0,2]$) $C(s,t) = 0$. The function is constant on L_1;
- Alongside L_3 ($s = 0$, $t \in [0,2]$) $C(s,t) = 0$. The function is constant on L_3;
- Alongside L_2 ($t = 2 - s$ and $s \in [0,2]$)

  $$C(s,t) = C(s,2-s) = s(2-s)(2+(2-s)-1) = s(2-s)$$

  So we need to optimise a function of one variable on $s \in [0,2]$.

$$C(s,t) = s(2-s), \qquad s = s,\ t = 2-s,\ s \in [0,2],$$

We proceed by

1. Making a list of:
   - Stationary points inside $[0,2]$:
     $$\frac{\mathrm{d}}{\mathrm{d}s}(s(2-s)) = 2-2s = 0 \Leftrightarrow s = 1;$$
   - (Extremal) Boundary points: $s = 0$, $s = 2$;
2. Evaluate the function in those points
   - For $s = 0$ (so $t = 2$): $C(0,2) = 0$;
   - For $s = 2$ (so $t = 0$): $C(2,0) = 0$;
   - For $s = 1$ (so $t = 1$): $C(1,1) = 1(2-1) = 1$

The maximum value of C on L_2 is 1 and it is attained at $(1,1)$;
The minimum value of C on L_2 is 0 and it is attained at $(0,2)$ & $(2,0)$.

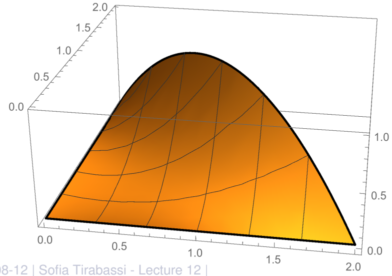
Evaluation and comparison

The list of points we have are

- $C(1,1) = 1$;
- $C(0,t) = 0$, $t \in [0,2]$;
- $C(s,0) = 0$, $s \in [0,2]$;
- $C(1/3, 1/3) = \frac{1}{3}\frac{1}{3}(\frac{1}{3} + \frac{1}{3} - 1) = -\frac{1}{27}$

As a result of the analysis above we have that:

- The maximum value of C on the domain D is 1 and it is attained at the points $(1,1)$;
- The minimum value of C on the domain D is $-1/27$ and it is attained at $(1/3, 1/3)$.



Lagrange Multipliers

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In some cases as above, we need to solve an optimisation problem $C(x, y)$ under some constrains of the form

$$g(x, y) = 0,$$

Example: In the previous example, we had to optimise $C(s, t)$ constrained to $s + t - 2 = 0$.

In that case, it was easy to write t as a function of s ($t = 2 - s$), plug that in the function $C(s, t)$ and treat that as a function in one variable. This is called the **substitution method**.

In general, one can't explicitly write one of the variables as a function of the other, but one can try by the **method of Lagrange Multipliers**

Lagrange multipliers method

Optimise the function $f(x, y)$ with the constraint $g(x, y) = 0$.

1. Construct the Lagrangian function

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda g(x, y),$$

2. The candidates for relative extrema occur at the solutions of

$$\begin{cases} \mathcal{L}'_x(x, y) = 0 \\ \mathcal{L}'_y(x, y) = 0 \\ g(x, y) = 0 \end{cases}$$

Be careful! The method only produce candidates for local extrema. In every problem we will need to make sure that global extrema exist.

The Economic Meaning of the Lagrange Multiplier λ

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What is λ Beyond the Algebra?

In mathematics, λ is just an auxiliary variable used to clear the absolute constraints. In economics, however, λ has a powerful structural definition: it is the **Shadow Price** (*skuggpris*) of the constraint.

The Mathematical Core: Let $f^*(c)$ be the optimal value of our objective function given a resource constraint capacity level c (e.g., $g(x, y) = c$). Then:

$$\lambda = \frac{\partial f^*(c)}{\partial c}$$

Real-World Economic Interpretations:

- **Consumer Theory:** If the constraint is your household budget, λ is the **Marginal Utility of Wealth**.
- **Production Theory:** If the constraint is a raw materials cap, λ tells a manager exactly how much additional net profit the firm would generate if the supplier allowed them to buy one more unit.

Example

Maximise $f(x,y) = x^2 - y^2$ subject to the constraint $2x + y - 5 = 0$.
Lagrange method: We construct the Lagrangian:

$$\mathcal{L}(x,y) = x^2 - y^2 + \lambda(2x + y - 5),$$

We (try to) solve the system of 3 equations an 3-unknowns (x, y, λ):

$$\begin{cases} 0 = \mathcal{L}'_x(x,y) = 2x + 2\lambda \\ 0 = \mathcal{L}'_y(x,y) = -2y + \lambda \\ 0 = 2x + y - 5 \end{cases} \Leftrightarrow \begin{cases} 0 = 2x \qquad + 2\lambda \\ 0 = \qquad -2y + \lambda \\ 5 = 2x + \quad y \end{cases}$$

From the first and second rows we have $x = -\lambda, \qquad y = \lambda/2,$
Plugging this in the third one yields

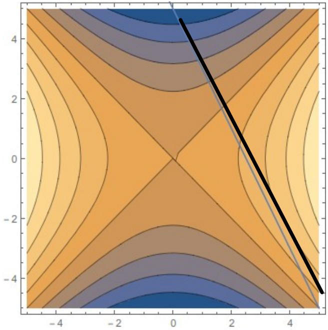
$$5 = 2(-\lambda) + (\lambda/2) = -3\lambda/2 \Leftrightarrow \lambda = -10/3$$

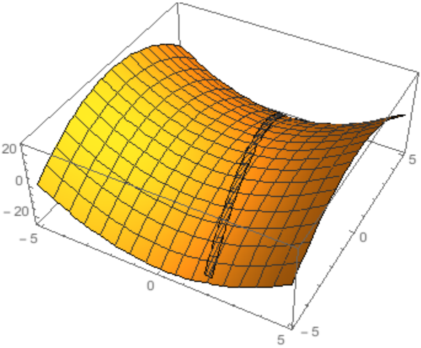
Hence

$$x = 10/3,\ y = -5/3 \quad f(10/3, -5/3) = (100 - 25)/9 = 25/3$$

Graphically

Looking at the level curves and the graph of f, we can see that the function, constrained to the line, has a global maximum at that point.





Example

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(2024) Let $F(x, y) = \sqrt{x^2 + y^2} - y^2 - 1$. Find its minimum and maximum on

$$D = \{x^2 + y^2 \leq 9, y \geq 0\},$$

Exam Workshop:

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Group Challenge: Parametrizing the Semi-Circle (4 Minutes)

Look at the 2024 exam problem on Slide 19: Optimize $F(x, y)$ on the domain:

$$D = \{x^2 + y^2 \leq 9,\ \ y \geq 0\}$$

Work with your team to map out your boundary list before computing:

1. **Piece 1 (The Flat Base):** Along the horizontal axis, the value of y is exactly 0. What is the valid domain range for the remaining variable x?
2. **Piece 2 (The Upper Rim):** Along the curved boundary edge, the equation is exactly $x^2 + y^2 = 9$. Substitute this identity directly into the function $F(x, y) = \sqrt{x^2 + y^2} - y^2 - 1$ to turn it into a simple single-variable expression!

Thank you for your attention!