MM3001

Lecture 13: Matrices and Linear Systems

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Review & Clarifications:

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Opening the Floor for Clarifications

Before we begin today's material on Matrices and Linear Systems, what questions do you have regarding last lecture's material on **Constrained Optimization, Boundary Hunting, or Lagrange Multipliers**?

60-Second Flash Recall: Warming Up Your Mathematical Brain

To help recall core mechanics while you look through your notes for questions:

- What is the economic intuition behind the Lagrange Multiplier parameter λ?
- When optimizing over a closed bounded region like a circle or triangle, why can't we ignore the boundary curves?

Lecture Goal and Outcome

Goals:

- Introduce the notion of vectors and matrices, their operations, and the connection to linear systems.
- Introduce the Gauss elimination method to solve linear systems.

Learning Outcome: At the end of the lecture you will be able to solve problem like the following:

Problem

The demand and supply equations of a good are given by

$$4P = -Q_d + 240, \qquad 5P = Q_s + 30,$$

where P is the price of the good, Q_d is the quantity demanded, and Q_s is the quantity supplied to the market.
Find the values of P, Q_s, Q_f such that the market is in equilibrium.

Why you should care

The Portfolio problem

Given a wealth W, we want to invest it in a collection of n risky assets $\{S_1, \ldots, S_n\}$: a fraction of our wealth, w_i say, in asset S_i.
Assuming that short selling is allowed (some of the w_i may be negative (borrow)).

> *How can we choose an optimal set of weights $\{w_1, ,,,, w_n\}$, so that our overall investment is likely to yield a promising return with minimal risk?*

To attack this problem, you need to learn several mathematical tools: Linear Algebra, Optimisation and Probability.

Lecture Plan

- Vectors & Matrices
- Matrix operations
- System of linear equations & Gaussian Elimination

Section 1
Vectors and Matrices

Vectors

We can represent a portfolio of *n*-stocks (or other assets) with a table of *n* rows and 1 column (vector) as

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

where x_i shares of stock $i = 1 \ldots, m$. We can define its transpose as

$$\vec{x}^T = \begin{pmatrix} x_1 & x_2 & \ldots & x_m \end{pmatrix}$$

Similarly, we can write

$$\vec{s}^T = \begin{pmatrix} s_1 & s_2 & \dots & s_n \end{pmatrix}$$

where s_i represents the price of a share of asset i under a particular economic scenario (or the)
The overall value of the portfolio under the given scenario is

$$s_1 x_1 + s_2 x_2 + \ldots + s_n x_n$$

Multiplication of a $(1 \times n)$ row vector and and $(n \times 1)$ column vector

$$\vec{s}^T \vec{x} := \begin{pmatrix} s_1 & s_2 & \dots & s_n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = s_1 x_1 + s_2 x_2 + \ldots + s_n x_n$$

Examples

- $\vec{s}^T = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$, $\vec{x}^T = \begin{pmatrix} -1 & 0 & 2 \end{pmatrix}$

Suppose now that we have m different scenarios, each with different price for share i:

$$\vec{s_j}^T = \begin{pmatrix} s_{j1} & s_{j2} & \dots & s_{jn} \end{pmatrix} \qquad j = 1, \dots, m,$$

We can represent all of them in a rectangular array of numbers with m rows and n columns ($m \times n$-Matrix):

$$S = \begin{pmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & & \ddots & \vdots \\ s_{m1} & s_{m2} & \dots & s_{mn} \end{pmatrix} = \begin{pmatrix} \vec{s_1}^T \\ \vec{s_2}^T \\ \vdots \\ \vec{s_m}^T \end{pmatrix}$$

The overall value of the portfolio for the m scenarios is given by the $m \times 1$-matrix

$$S\vec{x} := \begin{pmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & & \ddots & \vdots \\ s_{m1} & s_{m2} & \dots & s_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} \vec{s_1}^T \vec{x} \\ \vec{s_2}^T \vec{x} \\ \vdots \\ \vec{s_m}^T \vec{x} \end{pmatrix}$$

Examples

$$\begin{pmatrix} 3 & -7 \\ 2 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 7 \end{pmatrix} \qquad \begin{pmatrix} 3 & 2 \\ -2 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix},$$

Checking Dimensions in Financial Matrices

Think-Pair-Share: Vector Dimensions (4 Minutes)

On Slide 10, we computed the overall asset matrix multiplication $S\vec{x}$:

$$S\vec{x} = \begin{bmatrix} 3 & -7 \\ 2 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 7 \end{bmatrix}$$

Your Task with Your Bänkkamrat:

1. **Think (1 min):** Look at the output matrix. How many assets are we trading, and how many unique macro scenarios did the firm run?
2. **Pair (2 mins):** If we want to expand this simulation to track $k = 5$ separate asset portfolios under these same scenarios, what will the size dimensions of our new portfolio matrix X be?
3. **Share (1 min):** Why is it impossible to multiply an $m \times n$ matrix by an $m \times k$ matrix directly?

Suppose now, that we want to manage k-different portfolios, which we could represent now as a $n \times k$-Matrix

$$X := \begin{pmatrix} \vec{x}_1 & \vec{x}_2 \dots & \vec{x}_k \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1k} \\ x_{21} & x_{22} & \dots & x_{2k} \\ \vdots & & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nk} \end{pmatrix}$$

The overall value of the k different portfolios under m different scenarios is given by the $m \times k$ matrix

$$S \cdot X = \begin{pmatrix} Sx_1 & \dots Sx_k \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & & \ddots & \vdots \\ s_{m1} & s_{m2} & \dots & s_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1k} \\ x_{21} & x_{22} & \dots & x_{2k} \\ \vdots & & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nk} \end{pmatrix}$$

Multiplication of a $(m \times n)$ Matrix S and and $(n \times k)$ Matrix X

$$\begin{pmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & & \ddots & \vdots \\ s_{m1} & s_{m2} & \dots & s_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1k} \\ x_{21} & x_{22} & \dots & x_{2k} \\ \vdots & & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nk} \end{pmatrix}$$

gives a $m \times k$-matrix C where the element in the row i and column j is given by the matrix multiplication of the row i of S and the j column of X

$$c_{ij} = \begin{pmatrix} s_{i1} & s_{i2} & \dots & s_{in} \end{pmatrix} \cdot \begin{pmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{pmatrix} = s_{i1}x_{1j} + s_{i1}x_{2j} + \dots + s_{in}x_{nj}$$

Examples

$$\begin{pmatrix} 3 & -7 \\ 2 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 8 & 4 \\ 7 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 \\ -2 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ & \\ & \end{pmatrix},$$

Comparison of two matrices

Two matrices $(a_{ij})_{m\times n}$, $(b_{ij})_{k\times l}$ are equal if

1. Have the same order (same number of rows and columns) (i.e. $m = k$, $n = l$)
2. For all $i = 1, \dots, m$ and $j = 1, \dots, n$

$$a_{ij} = b_{ij},$$

Examples

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

The Matrix Equality Challenge

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Peer Instruction: Entry-by-Entry Parameter Matching (2 Minutes)

Let us solve the practice puzzle listed on Slide 15. We want to find the parameter vector $\vec{x}$ that enforces complete matrix equality:

$$\begin{bmatrix} 4 & x_2 \\ x_3 & 4 \end{bmatrix} = \begin{bmatrix} x_1 & 3 \\ 5 & x_4 \end{bmatrix}$$

Concept Check: What is the correct value matching for the parameters?

A) $x_1 = 4,\ x_2 = 5,\ x_3 = 3,\ x_4 = 4$

B) $x_1 = 4,\ x_2 = 3,\ x_3 = 5,\ x_4 = 4$

C) $x_1 = 3,\ x_2 = 4,\ x_3 = 4,\ x_4 = 5$

D) The system has no solution because positions cross.

Action: On my count of three, raise fingers matching your choice. Remember the rule on Slide 14: Matrices must match exactly position by position!

Sum of matrices

Whenever we have two matrices of the same order (same number of rows and columns) we can add them together term by term:

Given S, X two $m \times n$ matrices, we define $S + X$ as

$$\begin{pmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & & \ddots & \vdots \\ s_{m1} & s_{m2} & \dots & s_{mn} \end{pmatrix} + \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mk} \end{pmatrix}$$

$$= \begin{pmatrix} s_{11} + x_{11} & s_{12} + x_{12} & \dots & s_{1n} + x_{1n} \\ s_{21} + x_{21} & s_{22} + x_{22} & \dots & s_{2n} + x_{2n} \\ \vdots & & \ddots & \vdots \\ s_{m1} + x_{m1} & s_{m2} + x_{m2} & \dots & s_{mn} + x_{mn} \end{pmatrix}$$

Product by an scalar

Given an $m \times n$ matrix S, and a real value a we define aS as

$$\begin{pmatrix} as_{11} & as_{12} & \dots & as_{1n} \\ as_{21} & as_{22} & \dots & as_{2n} \\ \\ \vdots & & \ddots & \vdots \\ as_{m1} & as_{m2} & \dots & as_{mn} \end{pmatrix}$$

Examples:

$$3\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -7 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 3 \end{pmatrix} + \begin{pmatrix} -7 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 7 \end{pmatrix},$$

Transpose

If $\vec{x} = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}$ its transpose is $\vec{x}^T = \begin{pmatrix} x_{11} & x_{21} & \dots & x_{n1} \end{pmatrix}$

In general

$$X = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1k} \\ x_{21} & x_{22} & \dots & x_{2k} \\ \vdots & & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nk} \end{pmatrix} \text{ then } X^T = \begin{pmatrix} x_{11} & x_{21} & \dots & x_{n1} \\ x_{12} & x_{22} & \dots & x_{n2} \\ \vdots & & \ddots & \vdots \\ x_{1k} & x_{2k} & \dots & x_{nk} \end{pmatrix}$$

If A, B are $m \times n$-matrices, C $n \times k$-matrix, and $a \in \mathbb{R}$:

1. $(A^T)^T = A$
2. $(AC)^T = C^T A^T$
3. $(A+B)^T = A^T + B^T$
4. $(aA)^T = aA^T$

Section 2
Linear Systems

Linear systems: Motivation-I

The matrix S below represents the net profits to 3 stocks under 4 outcomes or scenarios:

$$S := \begin{pmatrix} -2 & 3 & 1 \\ -1 & 2 & 0 \\ 0 & -1 & 6 \\ 1 & -1 & -5 \end{pmatrix}$$

Can we invest in a way that produces the vector $\vec{v} = \begin{pmatrix} 0 & 0 & 0 & 5 \end{pmatrix}^T$?
How can we figure out the answer to this?
We look for a vector $\vec{x} = \begin{pmatrix} x & y & z \end{pmatrix}^T$ such that

$$\begin{pmatrix} -2 & 3 & 1 \\ -1 & 2 & 0 \\ 0 & -1 & 6 \\ 1 & -1 & -5 \end{pmatrix} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\text{Unknown}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \end{pmatrix} \Leftrightarrow \begin{cases} -2x + 3y + z = 0 \\ -x + 2y = 0 \\ -y + 6z = 0 \\ x - y - 5z = 5 \end{cases}$$

Linear systems: Motivation-II

The demand and supply equations of a good are given by

$$4P = -Q_d + 240, \qquad 5P = Q_s + 30,$$

where P is the price of the good, Q_d is the quantity demanded, and Q_s is the quantity supplied to the market.
At the point of intersection of the demand and supply curves (i.e. $Q_s = Q_p =: Q$), the market is said to be in equilibrium because the quantity demanded is equal to the quantity supplied.
How can we find that equilibrium point?
We need to find (P, Q) such that

$$\begin{cases} 4P + Q &= 240 \\ 5P - Q &= 30 \end{cases} \Leftrightarrow \begin{pmatrix} 4 & 1 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} 240 \\ 30 \end{pmatrix}$$

Linear systems

A linear system of m-(linear) equations and n-unknowns $x_1, \dots x_n$

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = b_1 \end{cases} \Leftrightarrow A\vec{x} = \vec{b},$$

where

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

A is called the coefficient matrix $\quad \vec{x}$ unknown $\quad \vec{b}$ independent term.

Taking the unknowns for granted, we can write the system in a compact form (Augmented coefficient matrix):

$$A\vec{x} = \vec{b} \Leftrightarrow \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \Leftrightarrow \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$$

Examples

$$\begin{pmatrix} 4 & 1 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} 240 \\ 30 \end{pmatrix} \Leftrightarrow \left(\begin{array}{cc|c} 4 & 1 & 240 \\ 5 & -1 & 30 \end{array}\right)$$

$$\begin{pmatrix} -2 & 3 & 1 \\ -1 & 2 & 0 \\ 0 & -1 & 6 \\ 1 & -1 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \end{pmatrix} \Leftrightarrow \left(\begin{array}{ccc|c} -2 & 3 & 1 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 6 & 0 \\ 1 & -1 & -5 & 5 \end{array}\right)$$

Simple systems $Ax = b$

The equation...

1. $1 \cdot x = 1$ has a unique solution $x = 1$.
2. The equation $0 \cdot x = 0$ is satisfied for all $x \in \mathbb{R}$. Has an infinite number of solutions.
3. $0 \cdot x = 1$ has no solution.

The system of equations...

1. $\left(\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & 30 \end{array}\right) \Leftrightarrow \begin{cases} x = 10 \\ y = 30 \end{cases}$ has only one solution.
2. $\left(\begin{array}{cc|c} 1 & 1 & 10 \\ 0 & 0 & 0 \end{array}\right) \Leftrightarrow \begin{cases} x + y = 10 \\ 0 \cdot x + 0 \cdot y = 0 \end{cases} \Leftrightarrow \begin{cases} x = 10 - t \\ y = t \end{cases} \quad t \in \mathbb{R},$
   The system has an infinite number of solutions.
3. $\left(\begin{array}{cc|c} 1 & 1 & 10 \\ 0 & 0 & 1 \end{array}\right) \Leftrightarrow \begin{cases} x + y = 10 \\ 0 = 1 \end{cases}$ has no solution.

Gauss elimination method

Strategy: Simplify the system by using the following elementary operations:

1. Exchange two equations (rows);
2. Multiplying equations (rows) by scalars;
3. Adding a multiple of one equation to another

Example

$$\left(\begin{array}{cc|c}5&4&25\\1&1&6\end{array}\right) \sim \left(\begin{array}{cc|c}1&1&6\\5&4&25\end{array}\right)\begin{matrix}-5\\+\end{matrix} \sim$$

$$\sim \left(\begin{array}{cc|c}1&1&6\\0&-1&-5\end{array}\right)\begin{matrix}\\ \mid\cdot(-1)\end{matrix} \sim \left(\begin{array}{cc|c}1&1&6\\0&1&5\end{array}\right)\begin{matrix}+\\-1\end{matrix} \sim \left(\begin{array}{cc|c}1&0&1\\0&1&5\end{array}\right)$$

The step of Gauss–Elimination

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1. We start from the first row. We choose a pivot (a non zero element) and we make all the elements under it 0.
2. We go to the second row. We choose a pivot and we make all the element under it 0.
3. ... we reiterate. We stop when either we have finished the rows or the next rows is all made by zeroes on the left of the bar.

Discussion of the output:

- If we have a row which is 0 on the left but not on the right the system has no solution.
- If all the row which are 0 on the left are so also on the right, then we count the number of non zero rows:
  - If the number of nonzero rows is equal the number of variables, then we have only one solution
  - if we have less nonzero rows than variables, then we have ∞-many solutions.

If we want to solve a bunch of systems with the same coefficients, but different independent terms

$$Ax = \vec{b}_1, \quad Ax = \vec{b}_2 \ldots \quad Ax = \vec{b}_k$$

we can run the method for all of them at the same time by writing

$$\left(\begin{array}{c|cccc} A & b_1 & b_2 & \ldots & b_k \end{array} \right)$$

Example Determine if any of following systems has any solution. If so, find all solutions.

1. $\begin{cases} 2x + y = 1 \\ 4x + 2y = 3 \end{cases}$
2. $\begin{cases} 2x + y = 1 \\ 4x + 2y = 2 \end{cases}$

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 1 \\ 4 & 2 & 3 & 2 \end{array} \right) \begin{array}{l} \urcorner^{-2} \\ \hookleftarrow_{+} \end{array} \sim \left(\begin{array}{cc|cc} 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right)$$

Exam Workshop:

Group Challenge: The Pivot Optimization Step (5 Minutes)

Let us set up the Gaussian elimination method for the 4-scenario historical stock portfolio system from before:

$$\left[\begin{array}{ccc|c} -2 & 3 & 1 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 6 & 0 \\ 1 & -1 & -5 & 5 \end{array}\right]$$

Work with your team to determine the most strategic opening move:

1. **Step 1:** Instead of working with fractions at the top row, what elementary row rule from Slide 24 lets us easily establish a clean 1 or -1 pivot element?
2. **Step 2:** If we swap Row 1 and Row 4, what operational multiplier clears the remaining elements under our new top position?
3. **Step 3:** Run the elimination path for the first column together!

Exercises

Determine if the systems of the motivational examples have any solution. If so, find the solution(s):

1. $\left(\begin{array}{cc|c} 4 & 1 & 240 \\ 5 & -1 & 30 \end{array}\right)$. $P = 30$, $Q = 120$
2. $\left(\begin{array}{ccc|c} -2 & 3 & 1 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 6 & 0 \\ 1 & -1 & -5 & 5 \end{array}\right)$

Thank you for your attention!