MM3001

Lecture 14: Matrices and Linear Systems-II

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Review & Clarifications: Lecture 13 Retrospective

Opening the Floor for Clarifications

Before we dive into determinants and matrix inverses today, what questions do you have regarding last lecture's introduction to **Matrix Multiplication or Gaussian Elimination**?

60-Second Flash Recall: Warming Up Your Linear Algebra Brain

To help recall core mechanics while you look through your notes for questions:

- If matrix A has size 3×2 and matrix B has size 2×4, what are the dimensions of the product matrix AB? Is BA defined?
- What are the three valid elementary row operations we can use during Gaussian Elimination?

Lecture Goal and Outcome

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Goals:

- Calculate determinants of Order 2 and 3 by cofactors.
- Determine whether a matrix is invertible or not and calculate its inverse.
- Determine the number of solutions of a linear system of equations.

Learning Outcome: At the end of the lecture you will be able to solve problem like the following:
Determine for which values of the parameter *a*, the system

$$\begin{cases} ax + y + 3z = 2 \\ 2x + y + az = 2 \\ 2x + y + 3z = a \end{cases}$$

has exactly one solution, no solution, or infinitely many solutions.

Inverse of a square matrix

A matrix with the same amount of rows and columns is called a **square matrix**.
We say that a $n \times n$ matrix A is **invertible** if there exists a matrix B such that

$$AB = BA = \mathbf{I_n}$$

where $\mathbf{I_n}$ is the identity matrix

$$\mathbf{I_n} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

If that matrix exists, it is unique, is called the **inverse** of A, and it is denoted by A^{-1}.

Determinant of a matrix

The linear equation $ax = b$ has a unique solution for all b if, and only if, a^{-1} exists (a is invertible), which is equivalent to $a \neq 0$.
Recall that a^{-1} is the only real number such that

$$a \cdot a^{-1} = a^{-1} \cdot a = 1,$$

For a given $a \in \mathbb{R}$, the 1×1 matrix $[a]$ is invertible if $\det[a] = a \neq 0$ and in that case

$$[a]^{-1} = [a^{-1}],$$

What can we say for $n = 2$?

Given a 2×2-matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ we define its determinant as $\det A = a_{11}a_{22} - a_{21}a_{12}$.

Example: If $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$ then $B = A^{-1}$ because

$$A \cdot B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B \cdot A$$

Moreover $\det A = 1 \cdot 3 - 2 \cdot 1 = 1, \quad \det B = ?$

A 2×2-matrix A is invertible, if and only if, $\det A \neq 0$

Note that if A is invertible, for all vector $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, the linear system of equations $Ax = b$ has a unique solution

$$Ax = b \Leftrightarrow A^{-1}Ax = A^{-1}b \Leftrightarrow x = A^{-1}b,$$

Example $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} =$
$\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

If $\det(A) \neq 0$, the linear system $Ax = b$ has a unique solution: $x = A^{-1}b$.
If $\det(A) = 0$, the linear system $Ax = b$ has either no solution, or an infinite number of solutions.
(we can use Gaussian elimination to find the solution(s))

How can we find the inverse?

If $\det A \neq 0$, we want to find B such that

$$A \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Leftrightarrow A \cdot \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \& A \cdot \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So, we need to solve two systems of Linear equations with the same independent coefficients (A)! So we can run the Gauss-elimination at the same time by writing:

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right) \begin{array}{l} \urcorner^{-2} \\ \hookleftarrow_{+} \end{array} \sim \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right) \begin{array}{l} \hookleftarrow^{+} \\ \lrcorner_{-1} \end{array} \sim \left(\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{array}\right)$$

Square matrices $n = 3$ and higher

Given a 3×3-matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ we define

$$\det A := a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

A similar definition can be given in any dimension $n \geq 3$. We will restrict ourselves to $n = 1, 2, 3$ in this course.

A $n \times n$-matrix A is invertible, if and only if, $\det A \neq 0$.
If $\det(A) \neq 0$, the linear system $Ax = b$ has a unique solution: $x = A^{-1}b$.
If $\det(A) = 0$, the linear system $Ax = b$ has either no solution, or an infinite number of solutions.

The same method as for $n = 2$ works in any dimension. E.g.

Determine whether the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$ is invertible, and if so, calculate its inverse:

$$\det A =$$

Strategic Cofactor Expansion

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Peer Instruction: Spotting the Path of Least Resistance (3 Minutes)

Look at the matrix A from our practice exercise on Slide 9:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

We need to compute $\det A$ using the cofactor expansion rules from Slide 8.

Concept Check: Which row or column is the most efficient choice to expand along?

A) Row 1, because it starts with 1.

B) Column 1, because all values are small integers.

C) Row 3, because the zero entry kills one entire sub-determinant.

D) Expanding along different rows yields different numerical answers.

$$
(\mathbf{A} \mid \mathbf{I}) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} -2 \quad -1 \\ + \\ + \end{array}
$$

$$
\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} \\ 2 \\ + \end{array}
$$

$$
\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) \begin{array}{l} \\ \\ \mid \cdot (-1) \end{array}
$$

$$
\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \begin{array}{l} + \\ + \\ 3 \quad -3 \end{array}
$$

$$
\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \begin{array}{l} + \\ -2 \\ \\ \end{array}
$$

$$
\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)
$$

Exam Workshop

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Think-Pair-Share: Sifting Singular Values (5 Minutes)

Let us set up our core learning outcome problem from Slide 3. We have a linear system with a variable parameter a:

$$\begin{cases} ax + y + 3z = 2 \\ 2x + y + az = 2 \\ 2x + y + 3z = a \end{cases} \implies A = \begin{bmatrix} a & 1 & 3 \\ 2 & 1 & a \\ 2 & 1 & 3 \end{bmatrix}$$

Your Task with Your Bänkkamrat:

1. **Think (2 mins):** Expand the determinant of the coefficient matrix A. Show that it simplifies to the expression: $\det A = (a-2)(3-a)$.
2. **Pair (2 mins):** Identify the two critical values of a where $\det A = 0$.What happens to the number of solutions when a avoids these two values?
3. **Share (1 min):** Why does $\det A \neq 0$ immediately guarantee a unique solution?

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For which values of c the system of equations below has a unique solution, infinitely many, or no solutions at all?

$$\begin{cases} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (c^2 - 14)z &= c + 2 \end{cases}$$

2019

Let A be the matrix $\begin{pmatrix} 2 & 0 & 2+k \\ 3 & 1 & 0 \\ k & 0 & -2 \end{pmatrix}$

1. Find $\det A$ as a function of k;
2. Show that A is always invertible for any value of k;
3. Find the minimal and maximal value $\det A$ can take, for $k \in [-2, 2]$.

Exam Workshop

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Group Challenge: Determinant Optimization (4 Minutes)

Look at the 2019 past exam problem on Slide 13. Matrix A is given by:

$$A = \begin{bmatrix} 2 & 0 & 2+k \\ 3 & 1 & 0 \\ k & 0 & -2 \end{bmatrix}$$

Work with your team to solve Part 3 of this exam profile:

1. **Step 1:** Expanding along Column 2 yields the determinant formula: $\det A = 1 \cdot [2(-2) - k(2+k)] = -4 - 2k - k^2$.
2. **Step 2:** Treat $\det A(k)$ as a downward-opening parabola function. Find its global vertex peak location using $k^* = -\frac{b}{2a}$.
3. **Step 3:** Evaluate the boundaries of the interval $k \in [-2, 2]$ to find the minimal and maximal values the determinant can structurally take!

Thank you for your attention!