Week 4 – Exercises

September 23, 2026

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Given a nonzero vector $(p_0, p_1, \ldots, p_n) \in \mathbb{K}^{n+1}$, we write $[p_0 : p_1 : \ldots : p_n] \in \mathbb{P}^n_{\mathbb{K}}$ for the corresponding point in projective space.

1 Tensor completion & Numerical solvers

A d-dimensional **tensor** of format $m_1 \times m_2 \times \ldots \times m_d$ is an array of numbers in $\mathbb{K}$ indexed by integer tuples $(i_1, i_2, \ldots, i_d)$, where $1 \leq i_j \leq m_j$ for all $j = 1, \ldots, d$. We write $\mathbb{K}^{m_1 \times \ldots \times m_d}$ for the set of all such tensors. Low-dimensional tensors are called: scalars for $d = 0$, vectors for $d = 1$, and matrices for $d = 2$.

Given d vectors $v \in \mathbb{K}^{m_1}, \ldots, v_d \in \mathbb{K}^{m_d}$,, we can produce a d-dimensional tensor via the **tensor product / outer product** $v_1 \otimes v_2 \otimes \ldots \otimes v_d$: The latter is a tensor whose entry indexed by $(i_1, i_2, \ldots, i_d)$ is $(v_1)_{i_1} \cdot (v_2)_{i_2} \cdots (v_d)_{i_d}$. This gives us a map

$$\mathbb{K}^{m_1} \times \ldots \times \mathbb{K}^{m_d} \longrightarrow \mathbb{K}^{m_1 \times \ldots \times m_d}, \; (v_1, \ldots, v_d) \longmapsto v_1 \otimes \ldots, \otimes v_d,$$

which is also know as the **Segre map**. Non-zero tensors in the image of this map are called **rank-one tensors**.

Example. For $d = 2$, rank-one tensors are precisely rank-one matrices.

For $d \geq 2$, not every non-zero tensor is of rank one.

Definition. Let $T \in \mathbb{K}^{m_1 \times \ldots \times m_d}$. The **rank** of T is the smallest $r \in \mathbb{Z}_{\geq 0}$ such that $T = T_1 + \ldots + T_r$ for some rank-one tensors $T_1, \ldots, T_r \in \mathbb{K}^{m_1 \times \ldots \times m_d}$.

Example. For $d = 2$, tensor rank is the usual matrix rank.

In the following two exercises, we suggest you use the software `HomotopyContinuation.jl`:

Exercise 1. *Let us consider 2×4 matrices. Matrices of rank at most one are parametrized by the map*

$$\begin{aligned} \mathbb{K}^2 \times \mathbb{K}^4 &\longrightarrow \mathbb{K}^{2 \times 4}, \\ (a, b) &\longmapsto a \otimes b. \end{aligned}$$

Note that $(\lambda a) \otimes (\frac{1}{\lambda} b) = a \otimes b$ for non-zero $\lambda \in \mathbb{K}$. To remove this scaling symmetry from the parametrization, we now assume that the first entry of a is 1; this parametrizes almost all rank-one matrices (namely those whose first row is non-zero).

a) *Perform the Jacobian check on the map with $a_1 = 1$ and conclude that it has generically finite fibers.*

b) *Find a rank-one completion of* $\left[\begin{smallmatrix}1&2&3&\\&&4&5\end{smallmatrix}\right]$ *(with $a_1 = 1$) using*

   (i) *homotopy continuation and*

   (ii) *monodromy.*

Exercise 2. *Let us consider three-dimensional tensors of format $3 \times 4 \times 2$. Tensors of rank at most two are parametrized by the map*

$$\begin{aligned}(\mathbb{K}^3 \times \mathbb{K}^4 \times \mathbb{K}^2)^2 &\longrightarrow \mathbb{K}^{3\times4\times2},\\ ((a_1, b_1, c_1), (a_2, b_2, c_3)) &\longmapsto a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2.\end{aligned}$$

As in the previous exercise, we assume now that the first entries of a_1, b_1, a_2 and b_2 are equal to 1 to remove the scaling symmetry from the parametrization (as before, this parametrizes almost all tensors of rank at most two).

a) *Perform the Jacobian check on the map with $(a_1)_1 = (b_1)_1 = (a_2)_1 = (b_2)_1 = 1$ and conclude that it has generically finite fibers.*

b) *Consider the partial tensor whose entries at $(i, j, 1)$ resp. $(i, j, 2)$ are:*

$$\begin{bmatrix}-32 & 72 & -104 & \\ & & 40 & 16\\ -24 & & 0 & \end{bmatrix} \quad \textit{resp.} \quad \begin{bmatrix} & -57 & -11 & -1\\ 10 & 27 & 1 & \\ & & & -7\end{bmatrix}.$$

   Complete it to a rank-2 tensor (with $(a_1)_1 = (b_1)_1 = (a_2)_1 = (b_2)_1 = 1$) via

   (i) *homotopy continuation and*

   (ii) *monodromy.*

c) *For your results from either homotopy continuation or monodromy:*

   (i) *Certify your solutions,*

   (ii) *verify that you found all solutions via the trace test,*

   and compare the running times of solving, certifying and completeness verification.

2 Discriminants

Exercise 3. *Compute the critical locus of the map $\varphi : X \to \mathbb{K}, (x, y) \mapsto x$, where $X := \{(x, y) \in \mathbb{K}^2 \mid x^2 + y^2 - 1 = 0\}$.*

Exercise 4. *Consider the following family (parametrized by a parameter p) of square polynomials systems in 2 variables x, y:*

$$f_1(x, y; p) = (x - 1) \cdot (x - 2) \cdot (x^2 + y^2 - 1),$$
$$f_2(x, y; p) = (y - 1) \cdot (y - p) \cdot (x^2 + y^2 - 1).$$

Recall that $N(p)$ denotes the number of regular zeros of the system at the parameter p. Show the following:

a) Solutions on the circle $x^2 + y^2 = 1$ are never regular.

b) $N(1) = 0$.

c) $N(p) = 3$ for $p \in \{0, \pm\sqrt{-3}\}$.

d) $N(p) = 4$ for $p \notin \{0, 1, \pm\sqrt{-3}\}$.

3 Basics: Zariski topology & Tangent spaces

Exercise 5. *Let $X \subseteq \mathbb{K}^n$ be an irreducible variety and let $Y \subsetneq X$ be a proper subvariety. Show that:*

a) The Zariski closure of $X \setminus Y$ equals X.

b) If a polynomial $f \in \mathbb{K}[x_1, \ldots, x_n]$ vanishes on $X \setminus Y$, it vanishes on all of X. [hint: use a)]

The next exercises illustrate that we need to use a "correct set" of defining equations to compute tangent spaces of varieties (namely, equations generating the vanishing ideal):

Exercise 6. *Consider the line which is the y-axis in the plane $\mathbb{C}^2$. It is the variety $Z(x)$, i.e., has the defining equation $x = 0$.*

a) Check that every point of the line is smooth and that its tangent space is the line itself.

An alternative description of the line is to use the defining equation $x^2 = 0$, i.e., to think of the line as the variety $Z(x^2)$.

b) Check that at the origin, the kernel of Jacobian of the equation $x^2 = 0$ is the whole plane.
[hence, this defining equation is not appropriate to compute tangent spaces]

Every variety X has a **decomposition into irreducible components**:

$$X = C_1 \cup \ldots \cup C_k.$$

This means that every C_i is an irreducible variety that is not contained in any C_j for $j \neq i$.

Exercise 7. *Let $\mathcal{F} := \{f_1, \ldots, f_s\} \subseteq \mathbb{K}[x_1, \ldots, x_n]$.*

a) *Show that $\langle \mathcal{F} \rangle = \{\sum_{i=1}^s h_i f_i \mid h_i \in \mathbb{K}[x_1, \ldots, x_n]\}$.*

b) *We consider the Jacobian matrix associated with $\mathcal{F}$:*

$$J_{\mathcal{F}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_s}{\partial x_1} & \cdots & \frac{\partial f_s}{\partial x_n} \end{bmatrix}$$

Let C be an irreducible component of the variety $Z(\mathcal{F}) \subseteq \mathbb{K}^n$. Show that at every smooth point p of C, the tangent space $T_p C$ is contained in the kernel of $J_{\mathcal{F}}(p)$.

[We've seen in the previous exercise that this containment can be strict.]

4 Back to discriminants

Exercise 8. *Let $f_1(\mathbf{x}) = \ldots = f_n(\mathbf{x}) = 0$ with $\mathbf{x} = (x_1, \ldots, x_n)$ be a square system of polynomial equations. Assume that the solution set is infinite. This means that it has at least one positive-dimensional irreducible component. For each such component C (of dimension ≥ 1), show the following:*

a) *All smooth points of C are non-regular solutions.* *[hint: 7b)]*

b) *All points of C are non-regular solutions.* *[use a) and 5b)]*