Week 6 – Exercises

October 7, 2026

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Given a nonzero vector $(p_0, p_1, \ldots, p_n) \in \mathbb{K}^{n+1}$, we write $[p_0 : p_1 : \ldots : p_n] \in \mathbb{P}^n_{\mathbb{K}}$ for the corresponding point in projective space.

1 Gröbner bases

Exercise 1. Compute a Gröbner basis (e.g., with `Macaulay2`) of

$$I := \langle x^2 + y^2 + z^2 - 1, x^2 + z^2 - y, x - z \rangle$$

with respect respect to the lexicographic monomial order induced by

a) $x > y > z$,

b) $z > x > y$.

Exercise 2. Compute a Gröbner basis of

$$\begin{gathered} I := \langle x_1x_2 + x_3x_4 + x_5x_6 + x_7x_8 - 1, x_1^2 + x_2^2 + x_3^2 - 1, x_2^2 + x_3^2 + x_4^2 - 1, \\ x_3^2 + x_4^2 + x_5^2 - 1, x_4^2 + x_5^2 + x_6^2 - 1, x_5^2 + x_6^2 + x_7^2 - 1, x_6^2 + x_7^2 + x_8^2 - 1, \\ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \rangle \end{gathered}$$

using

a) a graded reverse lexicographical ordering: GRevLex,

b) a lexicographical ordering (Lex).

Exercise 3. Show that there is just one monomial order on $\mathbb{K}[x]$, namely

$$\ldots > x^3 > x^2 > x^2 > x > 1.$$

2 Degrees

Exercise 4.

a) Show that every circle passes through the same 2 complex points at ∞. *(recall that a circle in $\mathbb{K}^2$ is defined by $(x-a)^2 + (y-b)^2 = r^2$ for some $a, b, r \in \mathbb{K}$, $r \neq 0$)*

b) Conclude that any 2 distinct circles intersect at 2 finite complex points (counted with multiplicity).

(if the 2 points are a single double point, this means that the circles are tangent to each other)

c) Show the following: For $\mathbb{K} = \mathbb{R}$, either both finite intersection points are real or both are non-real.

Exercise 5. Let $\mathcal{M}_1^{2\times 2} := \{M \in \mathbb{K}^{2\times 2} \mid \operatorname{rank}(M) \leq 1\}$ and consider the map

$$\varphi : \mathcal{M}_1^{2\times 2} \longrightarrow \mathbb{K}^3, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto (b, c, a-d)$$

(as in Week 1, Exercise 12). Show that:

a) $\deg(\varphi) = 2$;

b) for $\mathbb{K} = \mathbb{C}$ and $(b, c, \delta) \in \mathbb{C}^3$, we have that

$$|\varphi^{-1}(b, c, \delta)| = 2 \iff \frac{\delta^2}{4} + bc \neq 0 \quad \text{and}$$
$$|\varphi^{-1}(b, c, \delta)| = 1 \iff \frac{\delta^2}{4} + bc = 0$$

(in the latter case, the unique solution has multiplicity 2)

3 Back to Gröbner bases

The next 3 exercises will prove that a Gröbner basis for an ideal I indeed generates the ideal I.

Exercise 6. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $f_1, \ldots, f_k \in \mathbb{K}[\mathbf{x}]$. Show that

$$\langle f_1, \ldots, f_k \rangle = \{\sum_{i=1}^{k} p_i f_i \mid p_1, \ldots, p_k \in \mathbb{K}[\mathbf{x}]\}.$$

Exercise 7. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and let $\alpha_1, \ldots, \alpha_k, \beta \in \mathbb{Z}_{\geq 0}^n$. Show that

$$\mathbf{x}^\beta \in \langle \mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_k} \rangle \quad \Leftrightarrow \quad \exists i \in \{1, \ldots, k\} : \mathbf{x}^{\alpha_i} \mid \mathbf{x}^\beta.$$

(hint: use Exercise 4)

Exercise 8. Let $\mathbf{x} = (x_1, \ldots, x_n)$. Let $I \subseteq \mathbb{K}[\mathbf{x}]$ be an ideal, $>$ a monomial order on $\mathbb{K}[\mathbf{x}]$, and G a Gröbner basis for I with respect to $>$. Prove that

$$\langle G \rangle = I.$$

(hint: use Exercise 5)

Exercise 9. Let $c_0, \ldots, c_{d-1} \in \mathbb{K}$ and consider the $d \times d$ matrix

$$C := \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{d-1} \end{bmatrix}.$$

Show that $\det(xI_d - C) = c_0 + c_1 x + c_2 x^2 + \ldots + c_{d-1} x^{d-1} + x^d$.