

Commutative Algebra: an introductory course

Problem set 1

Submit by 28/09

You only need to submit solutions for any 6 of the following problems:

1. Let R be a ring. Show that if every R -module is free, then R is a field.
2. (a) Show that for all ideals I and J of a ring R there is a short exact sequence

$$0 \longrightarrow \frac{R}{I \cap J} \longrightarrow \frac{R}{I} \oplus \frac{R}{J} \longrightarrow \frac{R}{I+J} \longrightarrow 0.$$

- (b) Show that for each ideal I of a ring R and each $x \in R$ there is a short exact sequence

$$0 \longrightarrow \frac{R}{(I:x)} \xrightarrow{\cdot x} \frac{R}{I} \longrightarrow \frac{R}{I+(x)} \longrightarrow 0.$$

3. Let R be a ring. Show that $\text{Spec}(R)$ is quasi-compact. In other words, show that if $\mathcal{F} = \{U_\lambda\}_{\lambda \in \Lambda}$ is a family of open subsets of $\text{Spec}(R)$ such that $\text{Spec}(R) = \bigcup_{\lambda \in \Lambda} U_\lambda$, then there exist $U_{\lambda_1}, \dots, U_{\lambda_n} \in \mathcal{F}$ such that $\text{Spec}(R) = \bigcup_{i=1}^n U_{\lambda_i}$.
4. (a) Give an explicit description of $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/(n), \mathbb{Z}/(m))$, for any $n, m \geq 1$.
(b) Show that $\left(\frac{k[x,y]}{(xy)} \right)_x \cong k[x]_x$.
5. A ring R is reduced if, for every $r \in R, n \geq 1, r^n = 0$ implies $r = 0$. Show that R is reduced if and only if R_P is reduced for every $P \in \text{Spec}(R)$.
6. Give an example of a ring R and an R -module M such that M is Artinian but not Noetherian.
7. Let k be a field. Is $R = k[x, y, z]/(x^2, y^3, y^2x, z^2)$ Artinian? Yes. Why? Compute $\ell(R)$.
8. Let $(R, \mathfrak{m})$ be a local ring. A sequence $x_1, \dots, x_r \in R$ is said to be *crazy-independent* if any combination $a_1x_1 + \dots + a_rx_r = 0$ implies $a_i \in (x_1, \dots, x_r)$ for all $1 \leq i \leq r$. Let $I = (x_1, \dots, x_r)$, where $x_1, \dots, x_r \in R$ is a crazy-independent sequence.
 - (a) Show that $\{x_1, \dots, x_r\}$ is a minimal set of generators of I and that I/I^2 is a free R/I -module.
 - (b) Suppose I is such that R/I is Artinian. If $x_1 = yz$, show that

$$\ell(R/(x_1, \dots, x_r)) = \ell(R/(y, x_2, \dots, x_r)) + \ell(R/(z, x_2, \dots, x_r)).$$

9. Let R be a ring and $I_1, \dots, I_n$ be ideals of R satisfying $\bigcap_{j=1}^n I_j = (0)$ and such that R/I_j is Noetherian for all $1 \leq j \leq n$. Prove that R is Noetherian.

10. (a) Let M be a Noetherian R -module. Show that if $f : M \rightarrow M$ is a surjective homomorphism, then f is injective.
- (b) Give an example of a ring R and a surjective ring homomorphism $f : R \rightarrow R$ which is not injective.
- (c) Let now M be an Artinian R -module. Show that if $f : M \rightarrow M$ is an injective homomorphism, then f is surjective.
11. Let R be a Noetherian domain and $x \in R$ be a non invertible element. Prove that

$$\bigcap_{n=1}^{\infty} (x^n) = (0).$$

(Do not use stuff we haven't seen in class so far)

12. (*Challenging*)

(a) Let R be a ring such that

1. For every maximal ideal $\mathfrak{m}$, $R_{\mathfrak{m}}$ is Noetherian.
2. For every nonzero $x \in R$, the set of maximal ideals of R which contain x is finite.

Show that R is Noetherian.

(b) Take the (non Noetherian) ring $R = k[x_{11}, x_{21}, x_{22}, x_{31}, x_{32}, x_{33}, x_{41}, \dots]$. For each $i \geq 1$ let P_i denote the prime ideal $(x_{i1}, \dots, x_{ii})$ of R . Consider the multiplicatively closed subset

$$S = R \setminus (P_1 \cup P_2 \cup P_3 \cup \dots)$$

of R . Show that $S^{-1}R$ is Noetherian.

Please include at least one joke in your solutions!