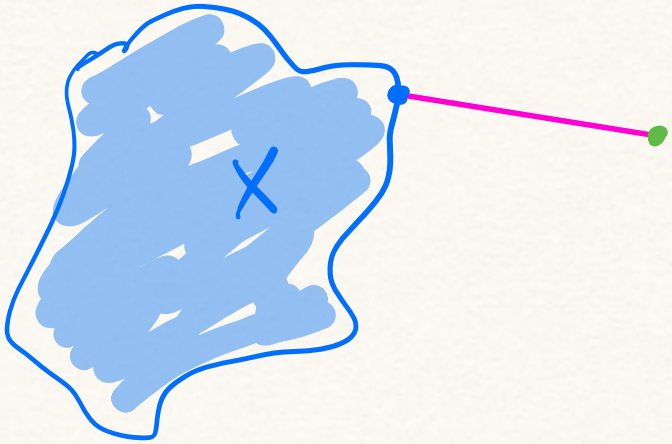
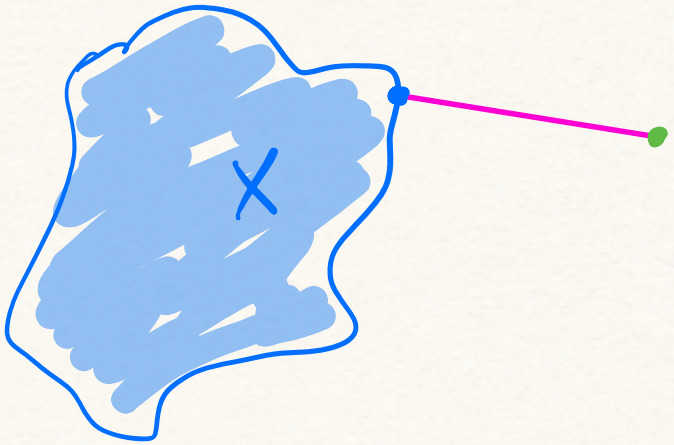
Algebraic Optimization Problems



Find closest point on an algebraic model from a given point.

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Assumptions for today: (1) $X \subseteq \mathbb{R}^n$ is an irreducible algebraic variety

(2) Euclidean metric

(3) closest point is smooth on X

Least-squares minimization on varieties

Given: $X \subseteq \mathbb{R}^n$ irreducible algebraic variety
$u \in \mathbb{R}^n$

We wish to solve: $$(\arg)\min_{x \in X} \|x-u\|^2 = (\arg)\min_{x \in X} \sum_{i=1}^{n} (x_i - u_i)^2$$

How would you solve this nonlinear & nonconvex optimization problem?

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↳ In general, the best you can do is:

① Find <u>all</u> critical points.

② Identify optimum among them.

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↳ by passing from $\mathbb{R}$ to $\mathbb{C}$ ☺

Lagrange multipliers theorem

$f: \mathbb{R}^n \to \mathbb{R}$ objective function

$g: \mathbb{R}^n \to \mathbb{R}^c$ constraints function

$\Big\} C^1$

Let p be optimal solution of

$$\min f(x) \text{ s.t. } g(x) = 0$$

and suppose $\operatorname{rank}(J_g(p)) = c$.

$J_g = \left[\frac{\partial g_i}{\partial x_j}\right]_{ij}$

$\Rightarrow$ There is a unique Lagrange multiplies $\lambda \in \mathbb{R}^c$ s.t.

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$$= \langle h_1, \dots, h_s \rangle$$

$c := \operatorname{codim}(X) := n - \dim(X) \quad \Rightarrow c \leq s$

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Fact: For all smooth $p \in X$, there is a Zariski-open neighborhood $p \in U \subseteq \mathbb{R}^n$ s.t. $X \cap U = Z(g_1, \dots, g_c) \cap U$.

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We call smooth p with this property critical points

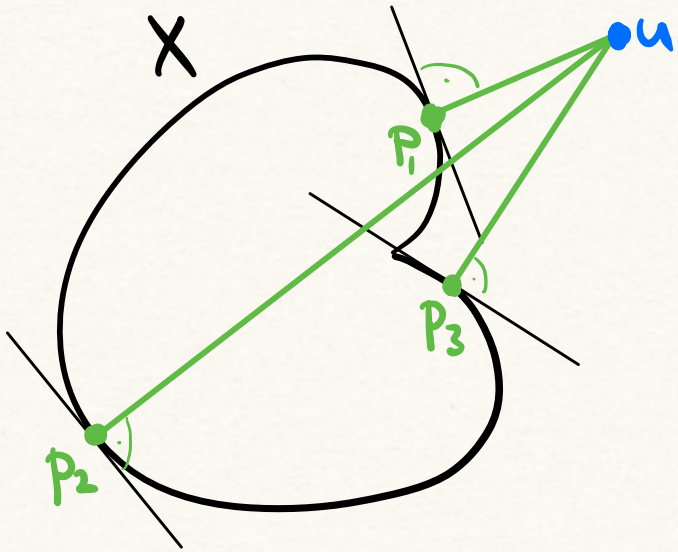
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Definition: A smooth point $p \in X$ is a critical point of $\bigstar$ if $\nabla f(p)$ is perpendicular to T_pX.

$\nabla f(p) = 2(p-u)$



How to compute critical points

Given: $X \subseteq \mathbb{R}^n$ irreducible algebraic variety
$u \in \mathbb{R}^n$

Let $\mathcal{H} = \{h_1, \dots, h_s\}$ be a generating set of $I(X)$.

At smooth points $p \in X$:
- $\operatorname{rank}(J_{\mathcal{H}}(p)) = \operatorname{codim}(X) =: c$
- $\ker(J_{\mathcal{H}}(p)) = T_p X$

↳ Corollary: For $p \in X$ smooth:

$p - u$ is perpendicular to $T_p X$ $\iff$ $\operatorname{rank}\begin{bmatrix} p^T - u^T \\ \hline J_{\mathcal{H}}(p) \end{bmatrix} = c$

augmented Jacobian matrix (of format $(s+1) \times n$)

How to compute critical points

specify u, e.g. $u_1 = 2$, $u_2 = 7$, ...

give generators of $I(X)$

augmented Jacobian

this ensures that $x \in X$ & $\operatorname{rank}(AJ(x)) \leq c$

```
n=2

QQ[x_1..x_n]
scan(n, i -> u_(i+1) = random(QQ))
I = ideal((x_1^2 + x_2^2 + x_2)^2 - (x_1^2 + x_2^2))



c = codim I
J = transpose jacobian I
xMinusU = matrix {apply(n, i -> x_(i+1)-u_(i+1))}
AJ = xMinusU || J
CritCond = I + minors(c+1,AJ)

Sing = radical ideal singularLocus I
Crit = saturate(CritCond,Sing)

dim Crit, degree Crit
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at this point, we still need to make sure that x is a smooth point generically!

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this computes

$$\{x \in X \text{ smooth} \mid \operatorname{rank}(AJ(x)) \leq c\}$$

these are the critical points :)

saturate(I_1, I_2) computes ideal of $\overline{Z(I_1) \setminus Z(I_2)}$

should be 0

number of critical points

Complexification

We now allow complex solutions to the polynomial equation system describing the critical points:

$$\left(I(X) + \text{minors}\left(c+1, \begin{bmatrix} x^T - u^T \\ \hline J_H(x) \end{bmatrix}\right)\right) : \text{Sing}(X)$$

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These are smooth points $p \in \mathbb{C}^n$ of $X(\mathbb{C})$ such that

($X(\mathbb{C})$: Zariski closure of X in $\mathbb{C}^n$)

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Definition: A smooth point $p \in X(\mathbb{C})$ is a complex critical point of ✡ if $(p-u)^T v = 0$ for all $v \in T_p X(\mathbb{C})$.

Finitely many critical points

Theorem: Let $X \subseteq \mathbb{R}^n$ be an irreducible algebraic variety.

For almost all $u \in \mathbb{R}^n$, the number of complex critical points of

$$(\arg)\min_{x \in X} \sum_{i=1}^{n} (x_i - u_i)^2$$

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What is the generic fiber of this map?

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What is the dimension of this?

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by fiber-dimension theorem:
this has dimension $(n - \dim X) + \dim X = n$

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$\pi_1 : (p,u) \mapsto p$

$X(\mathbb{C})$

over smooth p:

$\pi_1^{-1}(p) = p + T_p X(\mathbb{C})^{\perp}$ has dimension $n - \dim(X)$

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Theorem: Let $X \subseteq \mathbb{R}^n$ be an irreducible algebraic variety.

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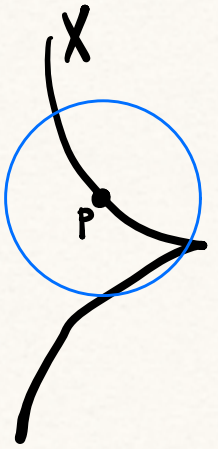
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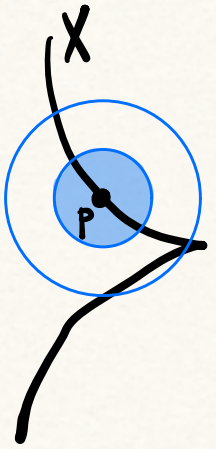
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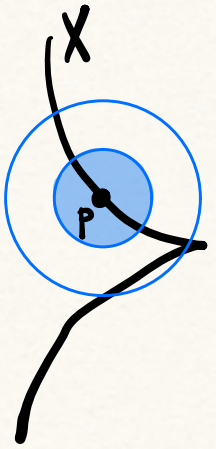
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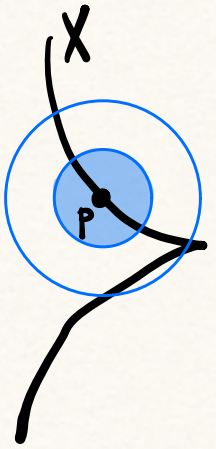
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5) Therefore, $B_{\frac{\varepsilon}{2}}(p) \subseteq \mathrm{im}(\pi_2)$. ($\dim_{\mathbb{R}} = n$)

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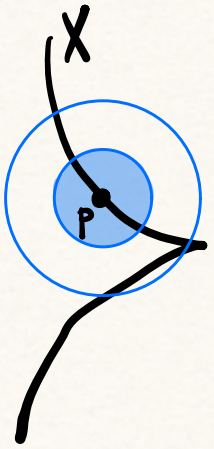
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5) Therefore, $\overline{B_{\frac{\varepsilon}{2}}(p)}$ ($\dim_{\mathbb{R}} = n$) $\subseteq \operatorname{im}(\pi_2)$.

6) $\overline{B_{\frac{\varepsilon}{2}}(p)} \subseteq \overline{\operatorname{im}(\pi_2)}$ (Zariski closure in $\mathbb{C}^n$)

Consider $\{(p,u) \in \mathbb{C}^n \times \mathbb{C}^n \mid p \in X(\mathbb{C}) \text{ is smooth}, (p-u)^T v = 0 \text{ for all } v \in T_p X(\mathbb{C})\}$

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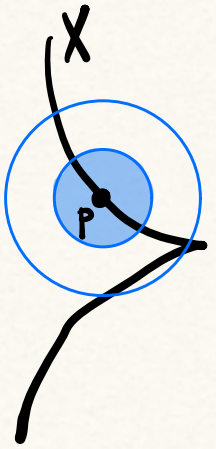
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5) Therefore, $\overline{B_{\frac{\varepsilon}{2}}(p)} \subseteq \mathrm{im}(\pi_2)$. ($\dim_{\mathbb{R}} = n$)

6) $\mathbb{C}^n = \overline{B_{\frac{\varepsilon}{2}}(p)} \subseteq \overline{\mathrm{im}(\pi_2)} \subseteq \mathbb{C}^n$ (Zariski closure in $\mathbb{C}^n$; Exercise)

◇

Finitely many critical points

Theorem: Let $X \subseteq \mathbb{R}^n$ be an irreducible algebraic variety.

For almost all $u \in \mathbb{R}^n$, the number of complex critical points of

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for almost all $u \in \mathbb{C}^n$, $\dim \pi_2^{-1}(u) = 0$ by fiber-dimension theorem

$\Leftrightarrow$

$0 < |\pi_2^{-1}(u)| < \infty$

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Exercise $\Rightarrow$ for almost all $u \in \mathbb{C}^n$: $0 < |\pi_2^{-1}(u)| < \infty$

$\Rightarrow$ for almost all $u \in \mathbb{R}^n$: $0 < |\pi_2^{-1}(u)| < \infty$

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Exercise ⟹ for almost all $u \in \mathbb{C}^n$: $0 < |\pi_2^{-1}(u)| < \infty$

for almost all $u \in \mathbb{R}^n$: $0 < |\pi_2^{-1}(u)| < \infty$

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Proof:

$$\text{Crit} := \left\{ (p,u) \in \mathbb{C}^n \times \mathbb{C}^n \;\middle|\; \begin{array}{l} p \in X(\mathbb{C}) \text{ is smooth,} \\ (p-u)^T v = 0 \text{ for all } v \in T_p X(\mathbb{C}) \end{array} \right\}$$

$$\pi_2 : \overline{\text{Crit}} \longrightarrow \mathbb{C}^n, \quad (p,u) \mapsto u$$

for almost all $u \in \mathbb{R}^n$: $0 < |\pi_2^{-1}(u)| < \infty$

$\pi_2^{-1}(u)$ contains complex critical points

Issue: $\pi_2^{-1}(u)$ could only consist of points in $\overline{\text{Crit}} \setminus \text{Crit}$ (and hence contain no complex critical points)

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Proof; Sketch:

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$$\pi_2 : \overline{\mathrm{Crit}} \longrightarrow \mathbb{C}^n, \quad (p,u) \mapsto u$$

not easy with our current tools

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$$\Rightarrow \dim(\pi_2(\overline{\text{Crit}} \setminus \text{Crit})) < n$$

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Proof; Sketch:

$$\mathrm{Crit} := \left\{ (p,u) \in \mathbb{C}^n \times \mathbb{C}^n \;\middle|\; \begin{array}{l} p \in X(\mathbb{C}) \text{ is smooth,} \\ (p-u)^T v = 0 \text{ for all } v \in T_p X(\mathbb{C}) \end{array} \right\}$$

$$\pi_2 : \overline{\mathrm{Crit}} \longrightarrow \mathbb{C}^n, \quad (p,u) \mapsto u$$

not easy with our current tools

$$\overline{\mathrm{Crit}} \setminus \mathrm{Crit} = \overline{\mathrm{Crit}} \cap (\mathrm{Sing}(X(\mathbb{C})) \times \mathbb{C}^n) \text{ is Zariski closed subset of } \overline{\mathrm{Crit}}$$

Since $\overline{\mathrm{Crit}}$ is irreducible (and $\overline{\mathrm{Crit}} \setminus \mathrm{Crit} \neq \overline{\mathrm{Crit}}$), $\dim(\overline{\mathrm{Crit}} \setminus \mathrm{Crit}) < \dim(\overline{\mathrm{Crit}}) = n$

$\Rightarrow \dim(\pi_2(\overline{\mathrm{Crit}} \setminus \mathrm{Crit})) < n$ $\Rightarrow$ for almost all $u \in \mathbb{C}^n$: $\pi_2^{-1}(u) \subseteq \mathrm{Crit}$

Finitely many critical points

Theorem: Let $X \subseteq \mathbb{R}^n$ be an irreducible algebraic variety.

For almost all $u \in \mathbb{R}^n$, the number of complex critical points of

$$(\arg)\min_{x \in X} \sum_{i=1}^{n} (x_i - u_i)^2$$ is finite (and positive).

we have proven this ☺ ↑ and this almost ☺

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$$\mathrm{Crit} := \left\{ (p,u) \in \mathbb{C}^n \times \mathbb{C}^n \;\middle|\; \begin{array}{l} p \in X(\mathbb{C}) \text{ is smooth,} \\ (p-u)^T v = 0 \text{ for all } v \in T_p X(\mathbb{C}) \end{array} \right\}$$

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$\Rightarrow$ for almost all $u \in \mathbb{R}^n$: $\pi_2^{-1}(u) \subseteq \mathrm{Crit}$ & $\pi_2^{-1}(u) \neq \emptyset$ ← from previous slide

□

Counting critical points

Theorem: Let $X \subseteq \mathbb{R}^n$ be an irreducible algebraic variety.

There is a positive integer E such that:

For almost all $u \in \mathbb{R}^n$, the number of complex critical points of

$$(\arg)\min_{x \in X} \sum_{i=1}^{n} (x_i - u_i)^2 \quad \text{equals } E.$$

Proof idea: compare parameter continuation theorem (PCT). □

Definition: The number E is called the Euclidean distance degree of X, denoted by $EDD(X)$.

Counting critical points

Remark 1: In practice, if you want to solve $(\arg)\min_{x \in X} \sum_{i=1}^{n} (x_i - a_i)^2$ given noisy observations $u \in \mathbb{R}^n$ by finding all real critical points: You'll know that you found all real critical points if you compute all complex ones and have found $\mathrm{EDD}(X)$ many!

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→ If there is no better algorithm for your particular variety X, the $\mathrm{EDD}(X)$ measures the complexity of finding the global minimum.

Counting critical points

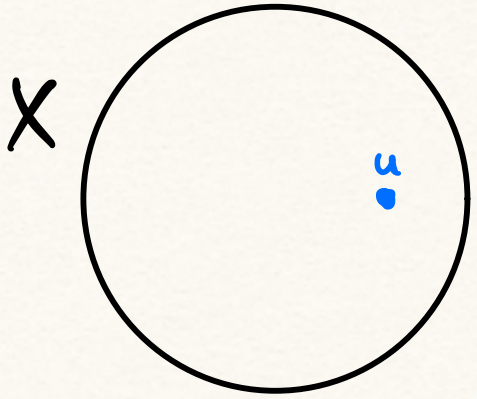
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→ If there is no better algorithm for your particular variety X, the $EDD(X)$ measures the complexity of finding the global minimum.

Remark 2: It follows also from the PCT that, for <u>all</u> $u \in \mathbb{R}^n$, the number of complex critical points is at most $EDD(X)$ or ∞.

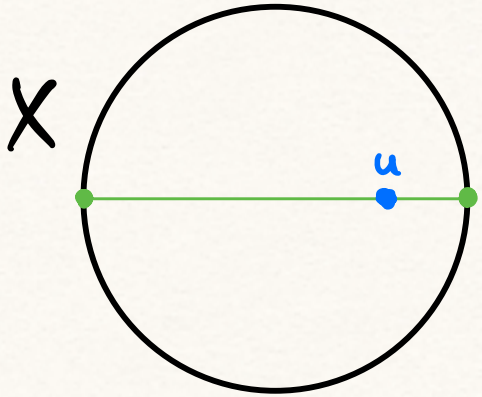
Examples



$x^2 + y^2 - 1 = 0$

What is the EDD(X)?

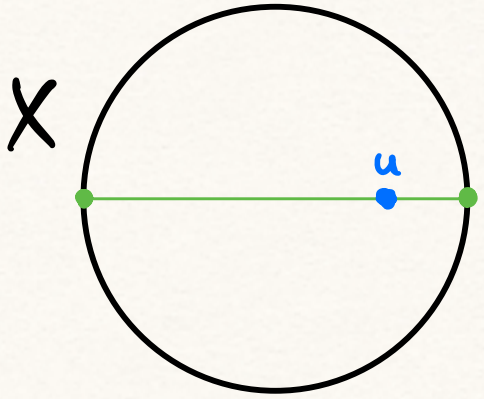
Examples



$x^2 + y^2 - 1 = 0$

What is the EDD(X)? 2

Examples

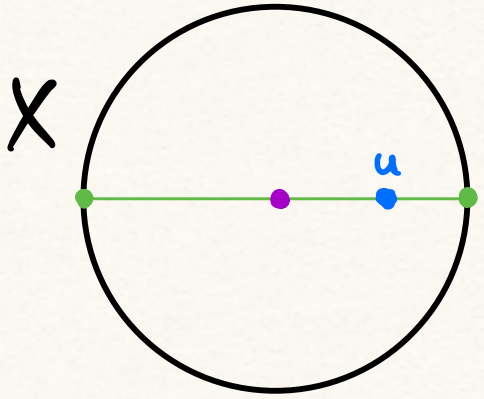


$x^2 + y^2 - 1 = 0$

What is the EDD(X)? 2

Are there $u \in \mathbb{R}^2$ for which the number of complex critical points is different from EDD(X)?

Examples

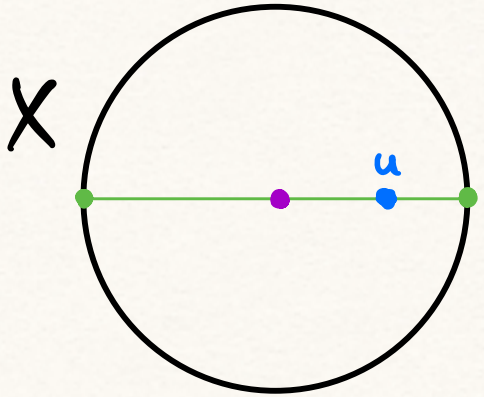


$x^2 + y^2 - 1 = 0$

What is the EDD(X)? 2

Are there $u \in \mathbb{R}^2$ for which the number of complex critical points is different from EDD(X)? Yes

Examples



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What is the EDD(X)? 2

Are there $u \in \mathbb{R}^2$ for which the number of complex critical points is different from EDD(X)? Yes

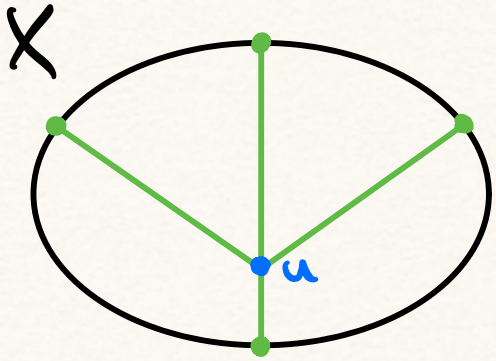
Definition: The Zariski closure of the set of $u \in \mathbb{C}^n$ for which

$$\left| \left\{ p \in X(\mathbb{C}) \mid \begin{array}{l} p \text{ is smooth point of } X(\mathbb{C}), \\ (p-u)^T v = 0 \text{ for all } v \in T_p X(\mathbb{C}) \end{array} \right\} \right| \neq EDD(X)$$

is called the **Euclidean distance (ED) discriminant** of X.

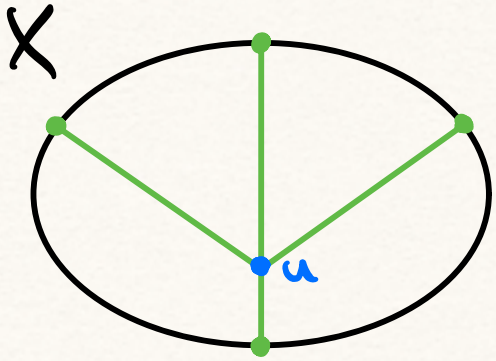
Example: The ED discriminant of the circle above is the origin.

Examples

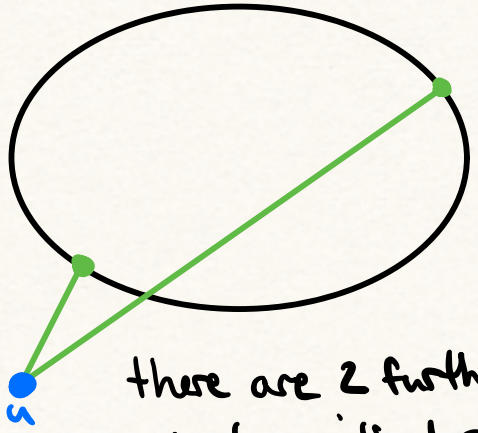


EDD(ellipse) = 4

Examples

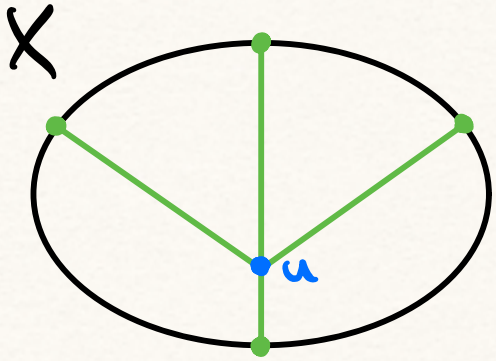


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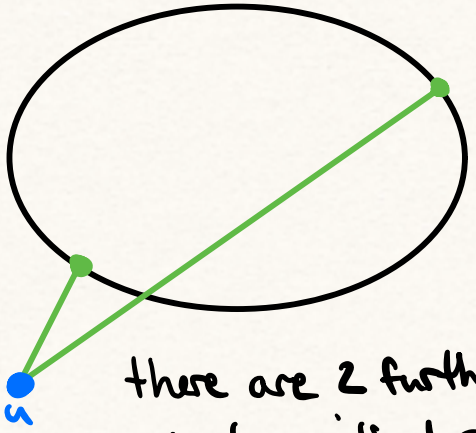


there are 2 further complex critical points

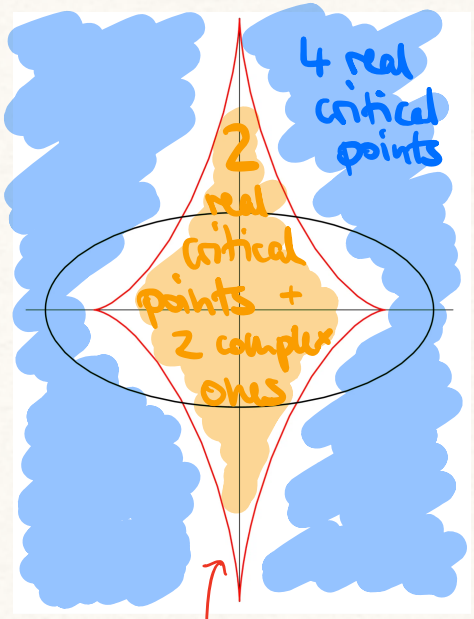
Examples



EDD(ellipse) = 4

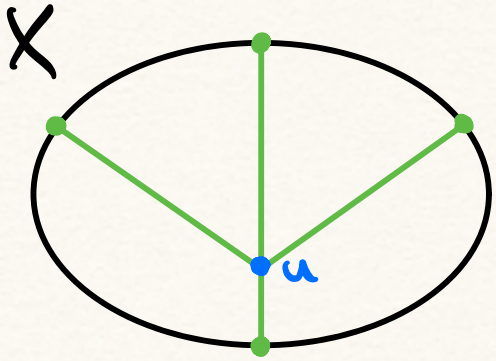


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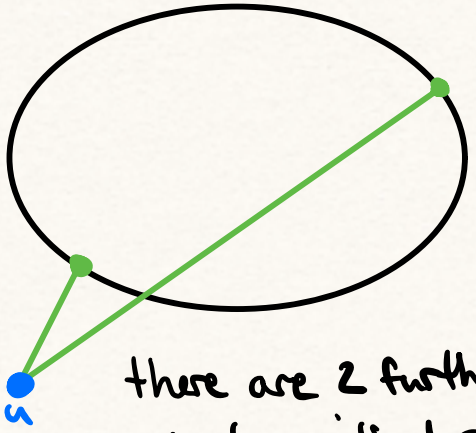


ED discriminant

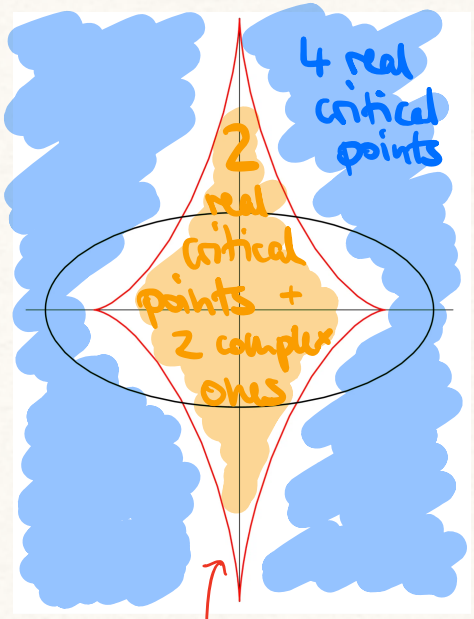
Examples



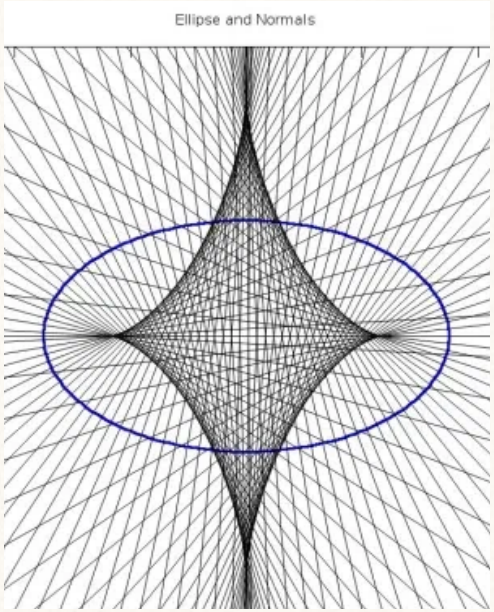
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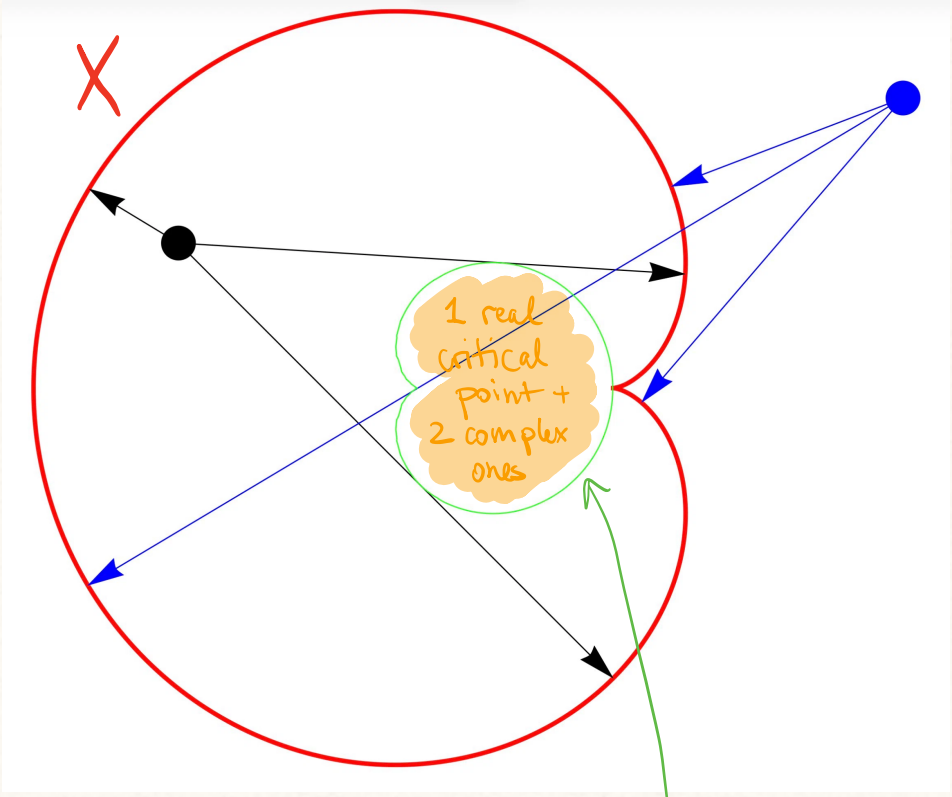


ED discriminant



When drawing many normals of the ellipse, the ED discriminant appears (it is tangent to all of them!)

Examples



EDD(cardioid curve) = 3

Low-Rank Matrix Approximation

$$X = \{ A \in \mathbb{R}^{m \times n} \mid \operatorname{rank}(A) \leq r \}$$

Given $U \in \mathbb{R}^{m \times n}$, solve: $\underset{A \in X}{\operatorname{argmin}} \|A - U\|_F^2$ ← Frobenius norm

$$= \sum_{ij} (a_{ij} - u_{ij})^2$$

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Assume: $m \leq n$

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$$U = T_1 \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_m \end{bmatrix} T_2, \quad \text{where } T_1 \in O(m),\ T_2 \in O(n),$$

$\sigma_1 \geq \sigma_2 \geq \dots \sigma_m \geq 0$.

singular values of U

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Eckart-Young Theorem:

A closest matrix on X to U is

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$\mathrm{EDD}(X) = ?$

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$\mathrm{EDD}(X) = \binom{m}{r}$

Low-Rank Matrix Approximation

$$X = \{ A \in \mathbb{R}^{m \times n} \mid \operatorname{rank}(A) \leq r \}$$

What is the ED discriminant of X?

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$$\sigma_1 \geq \sigma_2 \geq \cdots \sigma_m \geq 0.$$

($\sigma_1, \ldots, \sigma_m$: singular values of U)

Eckart-Young Theorem:

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Low-Rank Matrix Approximation

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What is the ED discriminant of X?

$U \in \mathbb{R}^{m \times n}$ is on the ED discriminant iff 2 of its singular values coincide.

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