

Commutative Algebra: an introductory course

Problem set 2

Submit by 12/10

You only need to submit solutions for any 6 of the following problems:

1. Let I and J be ideals of a ring R .
 - (a) Show that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J} = \sqrt{IJ}$.
 - (b) If R is Noetherian, show that there exists $n \geq 1$ such that $(\sqrt{I})^n \subseteq I$.
2. Let $R = k[x_1, \dots, x_n]$, where k is a field, and let P be an ideal of R generated by some of the variables $x_1, \dots, x_n$. Show that the powers of P are P -primary.
3. Let $R = k[x, y, z]$, where k is a field, and $I = (x^2, xy, xz, yz)$. Compute $\sqrt{I}$ and $\text{Ass}(R/I)$. Which associated primes of I are minimal primes and which are embedded primes? Compute also an irredundant primary decomposition of I .
4. Let R be a Noetherian ring and $x \in R$ be nonzerodivisor. Show that for all $n \geq 1$ we have that

$$\text{Ass}(R/(x)) = \text{Ass}(R/(x^n)).$$

5. Let Q be a P -primary ideal in a ring R . Show that if $x \notin Q$ then $(Q : x)$ is P -primary.
6. If I and J are ideals of a ring R , the *saturation of I with respect to J* is

$$(I : J^\infty) = \{r \in R : rJ^n \subseteq I \text{ for some } n \geq 1\}.$$

If $J = (x)$ is principal, then we denote $(I : (x)^\infty)$ by just $(I : x^\infty)$.

Suppose now R is Noetherian.

- (a) Show that $(I : J^\infty) = (I : J^n)$ for some $n \geq 1$.
- (b) If Q is a P -primary ideal, show that $(Q : J^\infty) = Q$ if $J \not\subseteq P$ and that $(Q : J^\infty) = R$ otherwise.
- (c) If $I = Q_1 \cap \dots \cap Q_r$ is a primary decomposition of I , show that

$$(I : J^\infty) = \bigcap_{J \not\subseteq \sqrt{Q_i}} Q_i.$$

- (d) Suppose I has no embedded primes and let $\text{Min}(I) = \{P_1, \dots, P_r\}$. Show that for each $1 \leq i \leq r$ there exists $x \in R$ such that the P_i -primary component of I is equal to $(I : x^\infty)$.
7. Let $(R, \mathfrak{m})$ be a Noetherian local ring and I a proper ideal of R which is not $\mathfrak{m}$ -primary. Show that $R/(I : \mathfrak{m}^\infty)$ has a nonzerodivisor which is not invertible.

(You might want to use the results from the previous problem)

8. Let $(R, \mathfrak{m})$ be a Noetherian local ring.

(a) If M is a finitely generated nonzero R -module, show that M is Artinian if and only if $\text{Ann}(M)$ is $\mathfrak{m}$ -primary if and only if $\text{Ass}(M) = \{\mathfrak{m}\}$.

(b) An element $f \in R$ is said to be a *party animal* if $f \notin P$ for every $P \in \text{Ass}(R) \setminus \{\mathfrak{m}\}$. Show that f is a party animal if and only if there exists $n \geq 1$ such that $\mathfrak{m}^n(0 : f) = 0$.

9. Show that an Artinian ring is isomorphic to a product of finitely many Artinian local rings.

(For this you might need the chinese remainder theorem for ideals - Theorem 1.1.11 in the notes)

10. Let R be a Noetherian reduced ring. Show that the total ring of fractions of R (i.e., the localization of R at the set of nonzerodivisors) is isomorphic to the product of finitely many fields.

11. (*Challenge?*) Let R be a Noetherian ring and I a proper ideal of R . Let $P \in \text{Ass}(R/I)$ (so that $P = (I : f)$ for some $f \notin I$). Set

$$Q = \prod_{\substack{P' \in \text{Ass}(R/I) \\ P \subsetneq P'}} P'$$

with the convention that $Q = R$ if there is no $P' \in \text{Ass}(R/I)$ such that $P \subsetneq P'$. Prove that, for any $f \in R$,

$$P = (I : f) \text{ if and only if } f \text{ is nonzero in } \frac{(I : P)}{(I : (P + Q^\infty))}$$

where $(I : (P + Q^\infty)) = \{r \in R : r(P + Q^n) \subseteq I \text{ for some } n \geq 1\}$.

Please be cool: include at least one joke in your solutions!