Commutative Algebra: an introductory course

Problem set 3

Submit by 02/11

You only need to submit solutions for any 6 of the following problems:

1. Let R be a $\mathbb{Z}$-graded ring and I and J be homogeneous ideals of R. Show that $(I : J)$ is also a homogeneous ideal of R.

2. Adapt the proof of the prime avoidance theorem (Theorem 1.2.14) to show the following graded version of prime avoidance: Let R be an $\mathbb{N}$-graded ring and I a homogeneous ideal of R generated by elements of positive degree. Let $P_1, \ldots, P_n$ be prime ideals of R such that $I \not\subseteq P_j$ for all j. Then there exists a homogeneous element $x \in I$ such that $x \notin P_1 \cup \cdots \cup P_n$.

3. Let R be an $\mathbb{N}$-graded ring and assume R is standard graded. Let $f \in R$ be a homogeneous element of degree d. Show that if the "multiplication by f" map $\cdot f : R \longrightarrow R$ is surjective in degree $N \geq 0$, then it is also surjective in degree n for all $n \geq N$. In other words, if the map $R_N \longrightarrow R_{N+d}$ given by $g \mapsto fg$ is surjective, then the map $R_n \longrightarrow R_{n+d}$ given by $g \mapsto fg$ is also surjective for all $n \geq N$.

4. Let R be a $\mathbb{Z}$-graded ring. Given an ideal I of R, denote by I^* the ideal generated by all the homogeneous elements in I (i.e., I^* is the largest homogeneous ideal contained in I). Show that if P is a prime ideal then P^* is also prime.

5. Let $R = k[x, y, z, w]$ with the standard grading, where k is a field. Compute the Hilbert series, Hilbert function, Hilbert polynomial and multiplicity of $R/(x^2, xy, yz)$. What is the dimension $R/(x^2, xy, yz)$?

6. Let I_1 be a homogeneous ideal of $R_1 = k[x_1, \ldots, x_n]$ and I_2 be a homogeneous ideal of $R_2 = k[y_1, \ldots, y_m]$ (both rings are standard graded, where k is a field). Consider the ideal I generated by $I_1 + I_2$ in the ring $R = k[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Show that

$$\mathrm{HS}_{R/I}(t) = \mathrm{HS}_{R_1/I_1}(t)\,\mathrm{HS}_{R_2/I_2}(t).$$

   Use this to compute the Hilbert series of $k[x_1, y_1, x_2, y_2, x_3, y_3]/(x_1y_1, x_2y_2, x_3y_3)$.

7. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Given a nonzero element $a \in R$, denote by $v(a)$ the unique n such that $a \in \mathfrak{m}^n \backslash \mathfrak{m}^{n+1}$. Set $v(0) = \infty$. Show the following:

   (a) $v(a + b) \geq \min\{v(a), v(b)\}$ for all $a, b \in R$.

   (b) $v(ab) \geq v(a) + v(b)$ for all $a, b \in R$. Here give an example where the inequality is strict.

   (c) If $\mathrm{gr}_{\mathfrak{m}}(R)$ is a domain (resp. reduced) then R is also a domain (resp. reduced).

   (d) Show that the converse of (c) is not true: give an example of a Noetherian local ring $(R, \mathfrak{m})$ for which R is a domain, but $\mathrm{gr}_{\mathfrak{m}}(R)$ is not even reduced.

8. Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. Similar to the previous problem, given a nonzero $x \in M$ denote by $v(x)$ the unique n such that $x \in \mathfrak{m}^n M \backslash \mathfrak{m}^{n+1} M$. Denote by $\text{in}(x)$ the homogeneous element $x + \mathfrak{m}^{v(x)+1} M \in \mathfrak{m}^{v(x)} M / \mathfrak{m}^{v(x)+1} M$ in $\text{gr}_{\mathfrak{m}}(M)$. Set $v(0) = \infty$ and $\text{in}(0) = 0$. Given a submodule N of M, define $\text{in}(N)$ to be the submodule of $\text{gr}_{\mathfrak{m}}(M)$ generated by all the $\text{in}(x)$ with $x \in N$. Show the following:

   (a) $\text{in}(N) = \displaystyle\bigoplus_{n=0}^{\infty} \frac{\mathfrak{m}^n M \cap N + \mathfrak{m}^{n+1} M}{\mathfrak{m}^{n+1} M} \cong \bigoplus_{n=0}^{\infty} \frac{\mathfrak{m}^n M \cap N}{\mathfrak{m}^{n+1} M \cap N}$.

   (b) $\text{gr}_{\mathfrak{m}}(M/N) \cong \text{gr}_{\mathfrak{m}}(M)/\text{in}(N)$.

   (c) *(optional)* If $N_1 \subseteq N_2$ are submodules of M, then $\text{in}(N_2/N_1) \cong \text{in}(N_2)/\text{in}(N_1)$ (where here we are using the inclusion $N_2/N_1 \subseteq M/N_1$ to compute $\text{in}(N_2/N_1)$).

9. Let's continue in the setting of the previous problem. A set of elements $\{x_1, \ldots, x_r\} \subseteq N$ is said to be a *standard basis* of N if $\text{in}(N) = (\text{in}(x_1), \ldots, \text{in}(x_r))$.

   (a) Show that a standard basis of N is a generating set of N.

   (b) The converse is not necessarily true: give an example of a Noetherian local ring R, an ideal I of R and a generating set of I which is not a standard basis of I.

10. Let $(R, \mathfrak{m})$ be a Noetherian local ring and I an ideal of R. Show that there exists $N > 0$ (depending only on R and I) such that the following holds: whenever J is an ideal of R satisfying $I + \mathfrak{m}^n = J + \mathfrak{m}^n$ with $n \geq N$, then there is a surjection

$$\text{gr}_{\mathfrak{m}}(R/I) \twoheadrightarrow \text{gr}_{\mathfrak{m}}(R/J).$$

11. *(Challenge!)* Let $(R, \mathfrak{m})$ be a Noetherian local ring and $N \subseteq M$ be finitely generated R-modules. Show that

$$\text{in}(\text{Ann}_R(N)) \subseteq \text{Ann}_{\text{gr}_{\mathfrak{m}}(R)}(\text{in}(N)) \subseteq \sqrt{\text{in}(\text{Ann}_R(N))}.$$

   Conclude that $\dim(N) = \dim(\text{in}(N))$.

Please be cool: include at least one joke in your solutions!