

Week 5 – Lab 1 Solutions

September 30, 2026

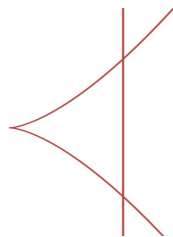
Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Given a nonzero vector $(p_0, p_1, \dots, p_n) \in \mathbb{K}^{n+1}$, we write $[p_0 : p_1 : \dots : p_n] \in \mathbb{P}_{\mathbb{K}}^n$ for the corresponding point in projective space.

1 Individual Part

Exercise 1. Determine all singular points of the plane curve defined by

$$(y^2 - x^3) \cdot (x - 1) = 0.$$

(2 points)



Solution:

We compute the Jacobian $[y^2 - x^3 - 3x^2(x - 1), 2y(x - 1)]$, which has, generically, rank 1 thus a point on the curve is singular if the rank of this matrix is 0, i.e. the matrix is the 0 matrix. The second entry tells us that for this to happen it is necessary that $2y(x - 1) = 0$, i.e. that either $y = 0$ or $x = 1$.

Starting with the case $y = 0$ we get from the first entry that $x^2(-x - 3(x - 1)) = 0$, i.e. that $x^2 = 0$ or $4x = 3$ giving us two points namely $(0, 0)$ and $(\frac{3}{4}, 0)$. In the case $x = 1$ the equation coming from the first entry of the Jacobian becomes $y^2 - 1 = 0$, i.e. $y = \pm 1$ giving the two points $(1, 1)$ and $(1, -1)$.

It remains to check which of these points are on the curve. We see that if $x = 1$ the 2nd factor of the defining equation is 0 and so the points are on the curve. For $(0, 0)$ we observe that the first factor becomes 0. Plugging in $(\frac{3}{4}, 0)$ we get $-\frac{27}{64} \cdot (-\frac{1}{4}) \neq 0$ so this point is not on the curve.

Hence, the singular points are $(0, 0)$, $(1, 1)$, and $(1, -1)$.

Exercise 2. Let $\mathbb{K} = \mathbb{C}$, and consider the parametrized polynomial equation $x^2 + px + q = 0$, where p, q are parameters and x is a variable. We denote the number of regular solutions at fixed parameters p, q by $N(p, q)$. Show that:

$$N(p, q) = \begin{cases} 2 & \text{if } q \neq \frac{p^2}{4}, \\ 0 & \text{if } q = \frac{p^2}{4}. \end{cases}$$

(2 points)

Solution:

A solution is regular if the Jacobian in the variables is invertible at the point if we insert the parameters and the solution. In the given scenario this means that a solution $\hat{x}$ to $x^2 + px + q$ is regular if $2\hat{x} + p \neq 0$. I.e. it is non-regular iff $\hat{x} = -\frac{p}{2}$. Using the pq-formula we get that the solution to our equation system are $\hat{x}_{\pm} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$. If $q \neq \frac{p^2}{4}$, the radicand is non-zero and so is the square-root; hence none of the two solutions equals $-\frac{p}{2}$ and thus they are both regular. If $\frac{p^2}{4} = q$, the radicand is 0, so there is only one solution, namely $-\frac{p}{2}$ which is non-regular as argued before. So in this case, there is no regular solution.

Exercise 3. Which of the following subsets of $\mathbb{R}^2$ are Zariski open, and why?

- a) $\mathbb{R}^2$
- b) $\mathbb{R}^2 \setminus \{(0, 0)\}$
- c) $Z(xy)$
- d) $\mathbb{R}_{>0} \times \mathbb{R}$

(2 points)

Solution:

- a) The complement of $\mathbb{R}^2$ (within $\mathbb{R}^2$) is $\emptyset = Z(1)$, hence the complement is Zariski closed and the set itself is Zariski open.
- b) The complement is $\{(0, 0)\} = Z(x, y)$, hence it is Zariski closed implying that the set $\mathbb{R}^2 \setminus \{(0, 0)\}$ is Zariski open.
- c) $Z(xy)$ is a proper subvariety of $\mathbb{R}^2$, which is irreducible. Hence, it follows from Week 4 – Exercise 5 that the Zariski closure of $\mathbb{R}^2 \setminus Z(xy)$ equals all of $\mathbb{R}^2$. Therefore, $\mathbb{R}^2 \setminus Z(xy)$ is not Zariski closed, and so $Z(xy)$ is not Zariski open.
- d) The complement of this set is $\mathbb{R}_{\leq 0} \times \mathbb{R} = \{(x, y) | x \leq 0\}$. It contains the open ball of radius 1 around $(-1, 0)$ and thus by Week 3 – Exercise 8, the only polynomial vanishing on $\mathbb{R}_{\leq 0} \times \mathbb{R}$ is the zero polynomial whose vanishing set is all of $\mathbb{R}^2$. Thus, $\mathbb{R}_{\leq 0} \times \mathbb{R}$ is not Zariski closed and hence $\mathbb{R}_{>0} \times \mathbb{R}$ is not Zariski open.

2 Team Work – Without Laptop

Exercise 4. a) Show that every line and every plane in $\mathbb{P}_{\mathbb{K}}^3$ intersect: Either the line is contained in the plane, or they meet in a unique point.¹

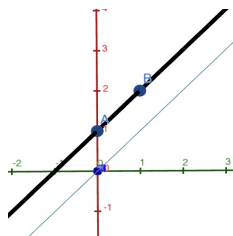
b) Fix the coordinates $[x : y : z : w]$ of $\mathbb{P}_{\mathbb{K}}^3$. Compute the point of intersection of the plane $Z(x - y)$ and the line spanned by $[0 : 1 : 0 : 1]$ & $[1 : 2 : 3 : 1]$.

(2 points)

Solution:

a) Let L be any line and P be any plane. Let A be a 2×4 matrix of rank 2 s.t. $L = \{x = [a : b : c : d] \mid Ax = 0\}$ and let v be a non-zero row vector with $P = \{x = [a : b : c : d] \mid vx = 0\}$. Then $L \cap P = \{x = [a : b : c : d] \mid \begin{pmatrix} A \\ v \end{pmatrix} x = 0\}$, where $\begin{pmatrix} A \\ v \end{pmatrix}$ is a 3×4 matrix of rank at least 2 and (obviously) at most 3. If the rank is 3, there is (up to scaling) a unique non-trivial vector in the kernel, which corresponds to exactly one intersection point in $\mathbb{P}^3$. If the rank is just 2, we see that $v \in \text{rowspan}(A)$, meaning that $L \subset P$.

b) $[0 : 1 : 0 : 1] \notin Z(x - y)$ and thus the line is not contained in the plane. So there exists a unique intersection point by a). We observe that $[1 : 2 : 3 : 1] - [0 : 1 : 0 : 1] = [1 : 1 : 3 : 0]$ is on the line and in $Z(x - y)$, thus it is the unique intersection point.



Now, let $\varphi : \mathbb{P}_{\mathbb{K}}^n \rightarrow \mathbb{P}_{\mathbb{K}}^n$ be a homogeneous polynomial map. We consider the fixed **affine chart** $\mathcal{A} := \{[p_0 : p_1 : \dots : p_n] \in \mathbb{P}_{\mathbb{K}}^n \mid p_n \neq 0\}$, and call the complement of $\mathcal{A}$ the **hyperplane at infinity**, denoted by $\mathbb{H} := \{[p_0 : \dots : p_{n-1} : 0] \in \mathbb{P}_{\mathbb{K}}^n\}$.

Exercise 5. Show the following:

For almost every² point $q \in \mathbb{P}_{\mathbb{K}}^n$, we have that $\varphi^{-1}(q) \subseteq \mathcal{A}$.

(2 points)

Solution:

Let $V := \overline{\varphi(\mathbb{H})}$ be the Zariski closure of the image of $\mathbb{H}$. Define $U := \mathbb{P}_{\mathbb{K}}^n \setminus V$. For any $u \in U$ it holds that $\varphi^{-1}(u) \subset \mathcal{A}$, since if there would be some $x \in \varphi^{-1}(u) \cap \mathbb{H}$, then $u = \varphi(x) \in \varphi(\mathbb{H}) \subseteq V$, contradicting that U is the complement of V . It remains to show that V is a proper subset of $\mathbb{P}^n$: This follows from the projective version of the fibre dimension theorem, stating that the dimension of V is at most the dimension of $\mathbb{H}$ which is $n - 1$ and hence $V \neq \mathbb{P}^n$.

Since the points in $\mathcal{A}$ look like $[p_0 : p_1 : \dots : p_{n-1} : 1]$, we see that $\mathcal{A}$ is isomorphic to $\mathbb{K}^n$ via

$$\alpha : \mathbb{K}^n \longrightarrow \mathcal{A}, (p_0, p_1, \dots, p_{n-1}) \longmapsto [p_0 : p_1 : \dots : p_{n-1} : 1].$$

¹A k -dimensional subspace of $\mathbb{P}_{\mathbb{K}}^n$ is the set of projective points that lie in the linear span of $k + 1$ fixed, linearly independent points. Alternatively, it is the zero locus of $n - k$ linearly independent linear equations.

²A property holds **for almost every** point of an irreducible variety X if there is a proper subvariety $\Delta \subsetneq X$ such that the property holds for all points in $X \setminus \Delta$.

Hence, if the image of φ is not contained in $\mathbb{H}$, we can transform the map $\varphi = [\varphi_0 : \varphi_1 : \dots : \varphi_n]$ into a **rational map** between affine spaces:

$$\tilde{\varphi} : \mathbb{K}^n \longrightarrow \mathbb{K}^n,$$

$$p = (p_0, p_1, \dots, p_{n-1}) \longmapsto \alpha^{-1}(\varphi(\alpha(p))) = \left(\frac{\varphi_0(\alpha(p))}{\varphi_n(\alpha(p))}, \dots, \frac{\varphi_{n-1}(\alpha(p))}{\varphi_n(\alpha(p))} \right),$$

which is well-defined at almost all points $p \in \mathbb{K}^n$.

Exercise 6. Assume that the Zariski closure of $\text{im}(\varphi)$ is $\mathbb{P}_{\mathbb{K}}^n$. Show that: For almost every point $q \in \mathbb{P}_{\mathbb{K}}^n$, we have that $q \in \mathcal{A}$ and

$$\varphi^{-1}(q) = \alpha(\tilde{\varphi}^{-1}(\alpha^{-1}(q))).$$

(1 point)

This means that, if φ has generically finite fibers (over $\mathbb{C}$) and we want to compute a generic fiber, we might as well do it with the rational map between the vector spaces $\mathbb{K}^n$. We will use this principle in the following computational task.

Solution:

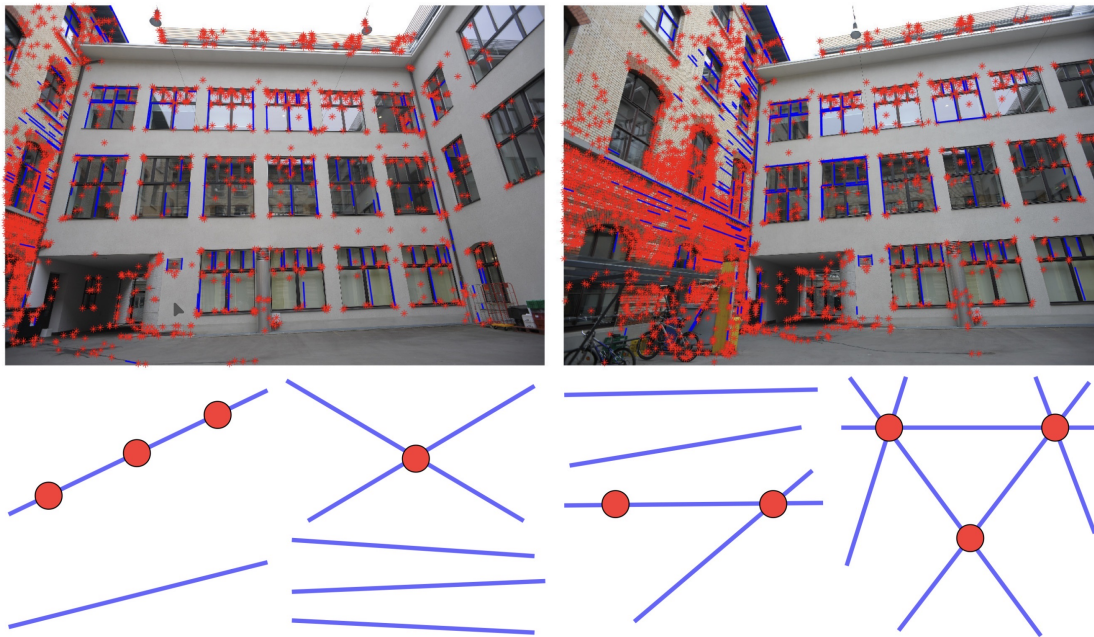
First we observe that $\overline{\text{im}(\varphi)} = \mathbb{P}^n$ implies that $\text{im}(\varphi) \not\subset \mathbb{H}$ as otherwise the closure would be contained in $\mathbb{H}$, too. So $\tilde{\varphi}$ is well-defined almost everywhere. Consider now the Zariski open set $\tilde{U} = U \cap \mathcal{A}$ (where U is taken as above). We can now be very careful and check that the fibres agree for all $q \in \tilde{U}$.

Let $p \in \varphi^{-1}(q)$ be arbitrary. Since $\tilde{U} \subset U$, we have $\varphi^{-1}(q) \subset \mathcal{A}$, i.e. $\alpha^{-1}(p)$ is well-defined and it suffices to show that $\alpha^{-1}(p) \in \tilde{\varphi}^{-1}(\alpha^{-1}(q))$. Applying $\tilde{\varphi}$ we get $\tilde{\varphi}(\alpha^{-1}(p)) = \alpha^{-1}(\varphi(\alpha(\alpha^{-1}(p)))) \stackrel{\alpha \text{ iso}}{=} \alpha^{-1}(\varphi(p)) = \alpha^{-1}(q)$, since $p \in \varphi^{-1}(q)$, which is well-defined as $q \in \mathcal{A}$.

Let $p \in \alpha(\tilde{\varphi}^{-1}(\alpha^{-1}(q)))$ be arbitrary, i.e. $\alpha^{-1}(p) \in \tilde{\varphi}^{-1}(\alpha^{-1}(q))$. We know that applying $\tilde{\varphi}$ yields $\alpha^{-1}(q)$, so $\alpha^{-1}(q) = \tilde{\varphi}(\alpha^{-1}(p)) = \alpha^{-1}(\varphi(\alpha(\alpha^{-1}(p)))) = \alpha^{-1}(\varphi(p))$, applying α to this equation yields $q = \varphi(p)$.

3 Team Work – With Laptop

Our goal is to recover the position and orientation of cameras from pictures that they taken of a 3D scene. We assume that, on the pictures, there have been points and lines indentified that some of the pictures have in common (there are many algorithms for this task developed by computer vision researchers), such as here:



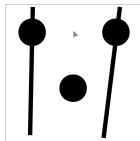
We then forget the actual pictures and only keep 2D arrangements of points and lines (like the ones shown above) that some of the pictures have in common. We now describe how to recover the 3D coordinates of the involved points and lines and the position and orientation of the involved cameras at an example:

A classical way of modeling a camera is as a 3×4 matrix

$$C = (R \mid t), \quad \text{where } R \in \text{SO}(3), t \in \mathbb{R}^3,$$

and to think of world and image points as points in the projective spaces $\mathbb{P}_{\mathbb{R}}^3$ and $\mathbb{P}_{\mathbb{R}}^2$, respectively. The camera takes a picture of a point $x \in \mathbb{P}_{\mathbb{R}}^3$ by producing the image point $Cx \in \mathbb{P}_{\mathbb{R}}^2$.

We consider the case where 3 pictures share a point-line arrangement looking like:



The variety of such arrangements in $\mathbb{P}_{\mathbb{R}}^k$ is denoted by

$$X_k := \{(x, y, z, L, K) \mid x, y, z \in \mathbb{P}_{\mathbb{R}}^k, L, K \subseteq \mathbb{P}_{\mathbb{R}}^k \text{ lines}, x \in L, y \in K\}. \quad (1)$$

Hence, our recovery problem means to compute a fiber under the following map:

$$\varphi : (\text{SO}(3) \times \mathbb{R}^3)^3 \times X_3 \longrightarrow X_2^3.$$

The map φ takes 3 cameras and a point-line arrangement in $\mathbb{P}_{\mathbb{R}}^3$ and produces the resulting 3 images, which are point-line arrangements in $\mathbb{P}_{\mathbb{R}}^2$.

The 3D points in the affine chart $\{(x_1 : x_2 : x_3 : x_4) \in \mathbb{P}_{\mathbb{R}}^3 \mid x_4 \neq 0\}$ are considered the finite points, while the points with $x_4 = 0$ are considered to be at infinity. Observe that the images of a finite point-line arrangement do not change when one simultaneously rotates or translates the arrangement and the cameras or scales the cameras. Formally, defining the *scaled special Euclidean group*

$$G := \left\{ \begin{pmatrix} R & t \\ 0 & \lambda \end{pmatrix} \mid R \in \text{SO}(3), t \in \mathbb{R}^3, \lambda \in \mathbb{R} \setminus \{0\} \right\},$$

we observe, for $g \in G$, the following:

1. When g transforms a camera $C \in \text{SO}(3) \times \mathbb{R}^3$, then Cg is still of the form $\text{SO}(3) \times \mathbb{R}^3$.
2. When g transforms cameras C to Cg and points $x \in \mathbb{P}_{\mathbb{R}}^3$ to $g^{-1}x$, then these transformations keep each fiber under the map φ invariant.

To mod out this group action out of the fibers of the map φ , we use a standard trick of computer vision engineers and mod it out of the camera triples: The following choice of representatives gives almost all camera triples modulo G :

$$\mathcal{C} := \{(I_3|0), (R|t), (S|u)) \mid R, S \in \text{SO}(3), t, u \in \mathbb{R}^3, t_3 = 1\},$$

where I_3 is the 3×3 identity matrix. As in Exercise 5, when computing generic fibers of φ , it suffices to consider almost all camera triples.

Similarly, we may assume that the 3D points x, y, z in (1) are finite. Then, we know from Exercise 4 that the line L (resp. K) in (1) is uniquely determined by x (resp. y) and its intersection point with the plane at infinity. Again, by the same principle as in Exercise 5, we may assume that the points at infinity are of the form $(l_1 : l_2 : 1 : 0)$ (i.e., such that the second to last entry is non-zero). Putting everything together, we restrict the domain of our map φ to

$$\mathcal{C} \times \tilde{X}_3, \quad \text{where}$$

$$\tilde{X}_3 := \{((x_1 : x_2 : x_3 : 1), (y_1 : y_2 : y_3 : 1), (z_1 : z_2 : z_3 : 1), (l_1 : l_2 : 1 : 0), (k_1 : k_2 : 1 : 0)) \in (\mathbb{P}_{\mathbb{R}}^3)^5\}.$$

Finally, we use the principle from Exercise 6 and model φ as a rational map:

1. When a camera C takes a picture of a finite point x , then we divide Cx by the last of the 3 coordinates.
2. When a camera C takes a picture of a line spanned by a finite point x and a point l at infinity, we model again the image line via its intersection point with the line $\{(a_1 : a_2 : 0) \in \mathbb{P}_{\mathbb{R}}^2\}$ at infinity and the point Cx . For that, we determine the point of the form $(a_1 : a_2 : 0)$ on the line spanned by Cx and Cl , and compute $\frac{a_1}{a_2}$.

The coordinates of the resulting rational map defined on $\mathcal{C} \times \tilde{X}_3$ are provided to you in the attached Julia code. There, the rotation matrices R and S are parametrized by Cayley's parametrization $\mathbb{R}^3 \rightarrow \text{SO}(3)$ (which parametrizes, again, almost all rotation matrices).

Exercise 7. Submit your code and screenshots of your output for the tasks a),c),d):

- a) Perform the Jacobian check.
- b) Conclude that the rational map has generically finite fibers.
- c) Compute a generic fiber (over $\mathbb{C}$) via monodromy.
- d) Compute a generic fiber (over $\mathbb{C}$) via homotopy continuation.

(4 points)