

Week 7 – Exercises

October 14, 2026

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Given a nonzero vector $(p_0, p_1, \dots, p_n) \in \mathbb{K}^{n+1}$, we write $[p_0 : p_1 : \dots : p_n] \in \mathbb{P}_{\mathbb{K}}^n$ for the corresponding point in projective space.

1 RANSAC

Exercise 1. We consider circles in $\mathbb{R}^2$ of a fixed radius $r > 0$.

- a) Show that, for distinct points $p, q \in \mathbb{R}^2$, there are at most 2 circles of radius r that pass through both p and q .
- b) Consider the following 10 points in $\mathbb{R}^2$:

$$\begin{array}{cccccccccc} p_0 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 \\ (7, 1) & (9, 7) & (-3, 1) & (-4, -5) & (2, 6) & (2, -4) & (1, 2) & (5, 5) & (8, -4) & (5, -3) \end{array}$$

Use RANSAC to find the circle of radius 5 that best fits these points, and determine which of the points you consider outliers.

Exercise 2. Say you have given data, where 30% of the data points are outliers. You perform RANSAC and in each repetition you select 5 data points to fit a model. How many repetitions do you need to be 99.9% sure that one of the selections did not contain any outliers?

2 Computer Vision

In today's lecture, we introduced a camera as a 3×4 matrix $C = [R|t]$, where $(R, t) \in \text{SO}(3) \times \mathbb{R}^3$. We identify the set of such matrices with the set $\text{SO}(3) \times \mathbb{R}^3$. When the camera C takes a picture of a point $x \in \mathbb{P}^3$, it computes $C \cdot x \in \mathbb{P}^2$.

Exercise 3. Show that

$$G := \left\{ \begin{bmatrix} S & u \\ 0 & \lambda \end{bmatrix} \mid S \in \text{SO}(3), u \in \mathbb{R}^3, \lambda \in \mathbb{R}, \lambda \neq 0 \right\}$$

is the largest subgroup of GL_4 such that, for all $C \in \text{SO}(3) \times \mathbb{R}^3$ and all $g \in G$, we have that $C \cdot g \in \text{SO}(3) \times \mathbb{R}^3$.

Exercise 4. Consider the *joint picture-taking map*

$$\begin{aligned}\varphi : (\mathrm{SO}(3) \times \mathbb{R}^3)^m \times (\mathbb{P}^3)^k &\longrightarrow ((\mathbb{P}^2)^k)^m, \\ (C_1, \dots, C_m, x_1, \dots, x_k) &\longmapsto ((C_1 x_1, \dots, C_1 x_k), \dots, (C_m x_1, \dots, C_m x_k)).\end{aligned}$$

Show that G acts on its fibers via

$$g \bullet (C_1, \dots, C_m, x_1, \dots, x_k) := (C_1 g, \dots, C_m g, g^{-1} x_1, \dots, g^{-1} x_k);$$

i.e., for all $(\mathbf{C}, \mathbf{x}) \in (\mathrm{SO}(3) \times \mathbb{R}^3)^m \times (\mathbb{P}^3)^k$ and all $g \in G$, we have that

$$\varphi(g \bullet (\mathbf{C}, \mathbf{x})) = \varphi(\mathbf{C}, \mathbf{x}).$$

Exercise 5. There is a Zariski closed proper subset $\Delta \subsetneq (\mathrm{SO}(3) \times \mathbb{R}^3)^2$ such that all camera pairs $(C_1, C_2) \in (\mathrm{SO}(3) \times \mathbb{R}^3)^2 \setminus \Delta$ can be written as

$$(C_1, C_2) = ([I_3|0] \cdot g, [R|\begin{smallmatrix} v \\ w \\ 1 \end{smallmatrix}] \cdot g)$$

for some $R \in \mathrm{SO}(3)$, $v, w \in \mathbb{R}$, and $g \in G$.

Exercise 6. We consider the map from Exercise 4 over the complex numbers in the case $m = 2$ and $k = 5$:

$$\varphi_{\mathbb{C}} : (\mathrm{SO}(3, \mathbb{C}) \times \mathbb{C}^3)^2 \times (\mathbb{P}_{\mathbb{C}}^3)^5 \longrightarrow (\mathbb{P}_{\mathbb{C}}^2)^5 \times (\mathbb{P}_{\mathbb{C}}^2)^5.$$

Similarly, we consider the “bad locus” from Exercise 5 over $\mathbb{C}$:

$$\Delta_{\mathbb{C}} \subsetneq (\mathrm{SO}(3, \mathbb{C}) \times \mathbb{C}^3)^2.$$

Check that the Zariski closure of $\varphi_{\mathbb{C}}(\Delta_{\mathbb{C}} \times (\mathbb{P}_{\mathbb{C}}^3)^5)$ is not all of $(\mathbb{P}_{\mathbb{C}}^2)^5 \times (\mathbb{P}_{\mathbb{C}}^2)^5$.

Hint: You can proceed as follows:

- a) Mod out only the rotation and the translation (i.e., the subgroup of G , where $\lambda = 1$) out of the fibers of $\varphi_{\mathbb{C}}$ in order to **always** turn the first camera into $[I_3|0]$.
- b) After this modding out, give an explicit description of $\Delta_{\mathbb{C}}$.
- c) Adapt the code from Lab 1 and perform the Jacobian check for the restricted map $\varphi_{\mathbb{C}}|_{\Delta_{\mathbb{C}} \times (\mathbb{P}_{\mathbb{C}}^3)^5}$ in order to see that its image has dimension less than 20.

Exercise 7. Now, in the case $m = 2$ and k arbitrary, consider the even further restricted map

$$\begin{aligned}\tilde{\varphi}_{\mathbb{C}} : \mathrm{SO}(3, \mathbb{C}) \times \mathbb{C}^2 \times (\mathbb{P}_{\mathbb{C}}^3)^k &\longrightarrow (\mathbb{P}_{\mathbb{C}}^2)^k \times (\mathbb{P}_{\mathbb{C}}^2)^k, \\ (R, v, w, x_1, \dots, x_k) &\longmapsto (([I_3|0]x_1, \dots, [I_3|0]x_k), \dots, ([R|\begin{smallmatrix} v \\ w \\ 1 \end{smallmatrix}]x_1, \dots, [R|\begin{smallmatrix} v \\ w \\ 1 \end{smallmatrix}]x_k)).\end{aligned}$$

- a) For $k \geq 6$, show that almost all fibers of $\tilde{\varphi}_{\mathbb{C}}$ are empty.
- b) For $k = 5$, adapt the code from Lab 1 and perform the Jacobian check in order to show that the image of $\tilde{\varphi}_{\mathbb{C}}$ has dimension 20.
- c) For $k = 5$, conclude from b) that almost all fibers of $\tilde{\varphi}_{\mathbb{C}}$ are non-empty and finite.
- d) For $k \leq 4$, conclude from c) that almost all fibers of $\tilde{\varphi}_{\mathbb{C}}$ have cardinality ∞ .
- e) For $k = 5$, adapt the code from Lab 1 and check that almost all fibers of $\tilde{\varphi}_{\mathbb{C}}$ have cardinality 20 using
 - (i) polynomial homotopy continuation,
 - (ii) monodromy.