

Week 8 – Exercises

October 21, 2026

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Given a nonzero vector $(p_0, p_1, \dots, p_n) \in \mathbb{K}^{n+1}$, we write $[p_0 : p_1 : \dots : p_n] \in \mathbb{P}_{\mathbb{K}}^n$ for the corresponding point in projective space.

1 Euclidean distance degree

Exercise 1. Use the `Macaulay2` code uploaded to the course website to compute the number of critical points for the Euclidean distance between a random (or your favorite) point $u \in \mathbb{R}^2$ and the curve $C \subseteq \mathbb{R}^2$, where

- a) $C = Z((x^2 + y^2 + y)^2 - (x^2 + y^2))$ is a cardioid curve,
- b) C is a circle,
- c) C is an ellipse.

Exercise 2. a) Use the `Julia` code uploaded to the course website to compute the critical points from Exercise 1 a)–c) via homotopy continuation.

- b) Why do we need to remove explicitly the singular locus of C in the `Macaulay2` code from Exercise 1, while this is not necessary for the `Julia` code in this exercise?

Exercise 3. Let $X \subseteq \mathbb{R}^6$ be defined by 3 random polynomials $f_1, f_2, f_3 \in \mathbb{R}[x_1, x_2, x_3, x_4, x_5, x_6]$ of degrees 3, 2, and 2, respectively. Compute the number of critical points for the Euclidean distance between a random point $u \in \mathbb{R}^6$ and X in the following ways and compare the run times:

- a) Use the `Macaulay2` code uploaded to the course website.¹
Hint: a random polynomial of degree k in a ring R can be produced via the following code:

```
sum apply(k+1, i -> random(i,R))
```

¹The execution of the code will probably be stopped in the online version. On a new MB-pro with 32GB RAM it took approximately 2h to terminate.

- b) Use homotopy continuation and adapt the `Julia` code uploaded to the course website.
- c) Introduce 3 new variables y_1, y_2, y_3 for the Lagrange multipliers and solve the following square system of polynomial equations: the 3 equations $f_1(x) = f_2(x) = f_3(x) = 0$ together with the 6 equations

$$(1, y_1, y_2, y_3) \cdot AJ(x) = 0,$$

where AJ is the augmented Jacobian matrix of f_1, f_2, f_3 (which has format 4×6).

- (i) Solve this in `Macaulay2` by adapting the code above.²
- (ii) Solve this in `Julia` with homotopy continuation by adapting the code above.

2 $\mathbb{R}$ is Zariski dense in $\mathbb{C}$

The following proves that the Zariski closure in $\mathbb{C}^n$ of a nonempty Euclidean open subset of $\mathbb{R}^n$ is the whole $\mathbb{C}^n$:

Exercise 4. Consider an open ball $\emptyset \neq \mathcal{B} \subseteq \mathbb{R}^n$ and a complex polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$. Show that, if $f(p) = 0$ for all $p \in \mathcal{B}$, then $f = 0$.
[hint: you may use Exercise 8a from Week 3 in the setting $\mathbb{K} = \mathbb{R}$]

Similarly, the following proves that, if a property holds for almost all points in $\mathbb{C}^n$, then it also holds for almost all points in $\mathbb{R}^n$:

Exercise 5. Let $\Delta \subsetneq \mathbb{C}^n$ be a proper Zariski closed subset of $\mathbb{C}^n$. Show that:

- a) $\Delta \cap \mathbb{R}^n$ is a Zariski closed subset of $\mathbb{R}^n$. (Given defining equations for Δ , what are defining equations for $\Delta \cap \mathbb{R}^n$?)
- b) $\Delta \cap \mathbb{R}^n \neq \mathbb{R}^n$.

3 More generators than codimension

Exercise 6. Consider the curve $C \subseteq \mathbb{P}_{\mathbb{C}}^3$ parametrized by

$$\varphi : \mathbb{P}_{\mathbb{C}}^1 \longrightarrow \mathbb{P}_{\mathbb{C}}^3, \quad [s : t] \longmapsto [s^3 : s^2t : st^2 : t^3].$$

- a) Writing $[x : y : z : w]$ for the coordinates of $\mathbb{P}_{\mathbb{C}}^3$, check that every point on the curve C satisfies the equations

$$y^2 - xz = 0, \quad z^2 - yw = 0, \quad yz - xw = 0.$$

²This code will probably not even terminate locally on a normal laptop as it requires too much memory.

- b) Prove that every point $[x : y : z : w]$ satisfying the above equations lies on the curve C .
- c) Choose any 2 of the 3 equations above and consider the ideal I generated by those 2 polynomials. Show that $Z(I)$ is the union of the curve C and some line. (Compute that line!)
- d) Prove that the vanishing ideal $I(C)$ cannot be generated by 2 equations (although the codimension of C is 2!)
[Hint: First, show that $I(C)$ contains no linear equation. Then, use that you have already shown in a) that the quadratic equations above are in $I(C)$.]

In fact, the vanishing ideal $I(C)$ is generated by the 3 quadratic equations above, which you can check with the following `Macaulay2` code:

```
R1 = QQ[s,t]
R2 = QQ[x,y,z,w]
kernel map(R1,R2,{s^3, s^2*t, s*t^2, t^3})
```