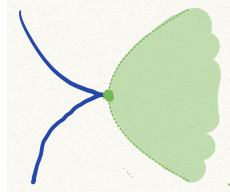


Week 9 – Exercises

October 28, 2026

1 A small example

- Exercise 1.** a) Compute a parametrisation of the curve bounding the Voronoi cell of $\{(x, y) \in \mathbb{R}^2 \mid y^2 + x^3 = 0\} = \{(-t^2, t^3) \mid t \in \mathbb{R}\}$ at the origin.
- b) Use Macaulay2 to find the defining equation of the bounding curve.



2 Low-Rank Matrix Approximation

Consider $X_{r,m \times n} := \{A \in \mathbb{R}^{m \times n} \mid \text{rank}(A) \leq r\}$ for $0 \leq r \leq m \leq n$.

Exercise 2. Show that $\dim X_{r,m \times n} = r(m + n - r)$.

Exercise 3. Let V be a real variety, $p \in V$ be a point, and $\gamma : (-\varepsilon, \varepsilon) \rightarrow V$ for some $\varepsilon > 0$ be a function with $\gamma(0) = p$. Show that if γ is differentiable at the origin, then $\gamma'(0) \in T_p V$.

Exercise 4. Let $A \in X_{r,m \times n}$ be of rank exactly r , and write $A = U_1 \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U_2$ for $U_1 \in O(m)$, $U_2 \in O(n)$, and $\Sigma \in \mathbb{R}^{r \times r}$ a diagonal matrix. Show that:

b) $T_A X_{r,m \times n} \supseteq \{U_1 \begin{bmatrix} B & C \\ D & 0 \end{bmatrix} U_2 \mid B \in \mathbb{R}^{r \times r}, C \in \mathbb{R}^{r \times (n-r)}, D \in \mathbb{R}^{(m-r) \times r}\}.$

(hint: use Exercise 3)

c) The inclusion in a) is an equality.

(hint: you may use Exercise 2 and that A is a smooth point of $X_{r,m \times n}$)

d) Under the Frobenius inner product $\langle M, N \rangle_F := \text{tr}(M^\top N)$, we have that

$$(T_A X_{r,m \times n})^\perp = \{U_1 \begin{bmatrix} 0 & 0 \\ 0 & E \end{bmatrix} U_2 \mid E \in \mathbb{R}^{(m-r) \times (n-r)}\}.$$

- Exercise 5.** a) Find a semialgebraic description (i.e., polynomial equations and inequalities) of $\mathcal{B} := \{E \in \mathbb{R}^{2 \times 2} \mid E^\top = E, \|E\|_{\text{op}} \leq 1\}$.
- b) Show that $\mathcal{B}$ is the convex hull of a circle and 2 points.
- c) What is the algebraic boundary of $\mathcal{B}$?

3 Neuromanifold boundary

Exercise 6. The neural network from Exercise 3 in Week 3 parametrizes a set of cubic polynomials, called the *neuromanifold* $\mathcal{M} \subseteq \mathbb{R}[x, y]_3$. Its Euclidean boundary $\partial\mathcal{M}$ is the set of polynomials $Ax^3 + Bx^2y + Cxy^2 + Dy^3$, whose discriminant $\delta := 18ABCD - 4B^3D + B^2C^2 - 4AC^3 - 27A^2D^2$ vanishes. Compute the Euclidean Distance Degree of $Z(\delta)$!