Commutative Algebra: an introductory course

Problem set 4

Submit by 16/11

You only need to submit solutions for any 6 of the following problems:

1. Given $n \geq 1$, find a ring R and an ideal I in R such that, for each $1 \leq i \leq n$, I has a minimal prime of height i.

2. In the ring $R = k[x, y, z, w, t]$, where k is a field, consider the ideal $I = (x^3, xy^2, yzw, xwt)$ Compute $\mathrm{ht}(I)$ and $\dim(R/I)$. Give a homogeneous system of parameters of R/I.

3. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Suppose that there exists a principal prime ideal P in R such that $\mathrm{ht}(P) \geq 1$. Prove that R is a domain.

4. Let $(R, \mathfrak{m})$ be a Noetherian local ring.

   (a) If $x \in R$ is not invertible, show that $\dim(R/(x)) \geq \dim(R) - 1$.

   (b) If $x \in R$ is a non invertible nonzerodivisor, show that $\dim(R/(x)) = \dim(R) - 1$.

   (c) Show that (a) can fail if R is not local. Hint: take $R = k[y, z] \times k[w]$.

5. Let $(R, \mathfrak{m})$ be a Noetherian local ring and J an $\mathfrak{m}$-primary ideal. Show that

$$\dim(\mathrm{gr}_{\mathfrak{m}}(R)) = \dim(\mathrm{gr}_J(R)) = \dim(R)$$

6. Let $A_\bullet$ and $B_\bullet$ be complexes of R-modules and $f, g : A_\bullet \longrightarrow B_\bullet$ be maps of complexes. Show that if f and g are homotopic, then f and g induce the same maps $H_n(A_\bullet) \longrightarrow H_n(B_\bullet)$ on homology.

7. Let $A_\bullet$ and $B_\bullet$ be complexes of R-modules. If $f : A_\bullet \longrightarrow B_\bullet$ and $g : B_\bullet \longrightarrow A_\bullet$ are homotopy equivalences, show that f and g are quasi-isomorphisms.

8. Let $(A_\bullet, d_A)$ and $(B_\bullet, d_B)$ be complexes of R-modules, with differentials d_A and d_B respectively, and $f : A_\bullet \longrightarrow B_\bullet$ be a map of complexes. The *mapping cone* $\mathrm{Cone}(f)$ of f is the complex $C_\bullet$ given by $C_n = A_{n-1} \oplus B_n$ and differential d_C given by $d_C(a, b) = (-d_A(a), d_B(b) + f(a))$ for all $a \in A_{n-1}$ and $b \in B_n$.

   (a) Show that $\mathrm{Cone}(f)$ is in fact a complex.

   (b) Show that there exists a short exact sequence of complexes

$$0 \longrightarrow B_\bullet \xrightarrow{\iota} \mathrm{Cone}(f) \xrightarrow{p} A_\bullet[-1] \longrightarrow 0$$

   where ι is the inclusion given by $\iota(b) = (0, b)$ and p is the projection given by $p(a, b) = a$. Here $A_\bullet[-1]$ is the complex given by $(A_\bullet[-1])_n = A_{n-1}$ and differential $d_{A_\bullet[-1]} = -d_{A_\bullet}$.

(c) From (b) there is a long exact sequence

$$\cdots \longrightarrow H_{n+1}(\mathrm{Cone}(f)) \longrightarrow H_n(A_\bullet) \xrightarrow{\partial} H_n(B_\bullet) \longrightarrow H_n(\mathrm{Cone}(f)) \longrightarrow H_{n-1}(A_\bullet) \longrightarrow \cdots$$

where ∂ is the connecting homomorphism. Show that ∂ is the map induced by f on homology.

(d) Conclude that f is a quasi-isomorphism if and only if $\mathrm{Cone}(f)$ is exact.

9. A short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules is said to *split* if it is isomorphic to

$$0 \longrightarrow A \xrightarrow{i} A \oplus C \xrightarrow{p} C \longrightarrow 0 \ ,$$

where i is the inclusion of the first component and p is the projection onto the second component. Show that the following are equivalent.

(i) The exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ splits.

(ii) There exists an R-module homomorphism $q : B \longrightarrow A$ such that $qf = \mathrm{id}_A$.

(iii) There exists an R-module homomorphism $h : C \longrightarrow B$ such that $gh = \mathrm{id}_C$.

10. *(Challenge?)* Let $(R, \mathfrak{m})$ be a Noetherian local ring of Krull dimension d and let $a_1, \ldots, a_d$ be a system of parameters of R. Denote by $J = (a_1, \ldots, a_d)$ the $\mathfrak{m}$-primary ideal they generate. Let $f(t_1, \ldots, t_d) \in R[t_1, \ldots, t_d]$ be a homogeneous polynomial of degree r. Assume that

$$f(a_1, \ldots, a_d) \in J^{r+1}.$$

Show that all the coefficients of f are in $\mathfrak{m}$.

Please be cool: include at least one joke in your solutions!