Topics for presentations

Commutative Algebra: an introductory course

Please pick a topic which is new for you. All that is required is that you do a 30 min presentation about the topic that you choose. 30 min is short, so focus on giving the main definitions and results, while trying to provide some intuition for the topic. Also, try to include a proof (or at least a sketch of one).

Some of the topics I propose are vast and thus allow for two or more separate presentations by different people. It is also possible for two people to present together in a 1 hour presentation, giving more time to present the topic in more detail.

1. **Local cohomology:** Introduce the local cohomology modules $H^i_I(-)$ as the right-derived functors of the functor Γ_I and present some of their basic properties. Talk about the vanishing and non-vanishing theorems: [BS13, Theorems 6.1.2, 6.1.4 and 6.2.7]. Perhaps also talk about how the local cohomology modules can be computed as the cohomology of Cech complexes.

   Reference: [BS13].

   This is a particularly good topic for two people to present together in a 1 hour presentation.

2. **Flatness:** Flat and faithfully flat modules. Present some characterizations of flatness (like [Mat86, Theorem 7.7] and the characterization via vanishing of Tor modules [Mat86, Theorem 7.8]). Consequences of flatness.

   References: [Mat86, Chapter 7].

3. **Completion:** Introduce the completion of a ring at an ideal I and talk about some of it's properties, for instance the fact that it is flat. Cohen's structure theorem.

   References: [Mat86, Chapter 8], [AM69, Chapter 10] and [Eis13, Chapter 7].

4. **Injective modules:** Go into more detail on the theory of injective modules. There are a few possibilities here. Presenting a proof that every module can be embedded into an injective module ([BH98, Theorem 3.1.8]). Injective dimension and modules with finite injective dimension ([BH98, Theorem 3.1.17]). Essential extensions and injective hulls. Decomposing and injective module as a direct sum of indecomposable injective modules. Bass numbers ([BH98, Section 3.2]).

   Reference: [BH98, Sections 3.1 and 3.2].

   There is enough here for two or three presentations.

5. **Initial ideals and Gröbner basis:** Talk about monomial ideals, monomial orders, initial ideals and Buchberger's algorithm. Show that the initial ideal has the same Hilbert function as the original ideal.

   Reference: [HH11, Chapter 1 and 2] and [Eis13, Chapter 15].

   This can be a 1 hour presentation by two people.

6. **Macaulay's inverse system:** Talk about Poincaré duality and the inverse system. Characterize Artinian Gorenstein graded k-algebras as quotients by the annihilator of a single homogeneous form.

   References: A bit hard to find, but I can provide some notes on it upon request.

7. **Going-up and going-down theorems:** Talk a bit about the integral closure and these two theorems.

   References: [HS06, Section 2.2], [Mat86, Section 9] and [AM69, Chapter 5].

8. **Valuations, DVRs and Dedekind rings:** Cover some of the definitions and results of [Mat86, Sections 10 and 11]

9. **Assymptotic behaviour of the associated primes of powers of ideals:** Present a sketch or a full proof of the following result: if R is a Noetherian ring and I an ideal of R, then the set $\mathrm{Ass}(R/I^n)$ eventually stabilizes, as well as the set $\mathrm{Ass}(I^n/I^{n+1})$.

   Reference: Section 6.3 in the notes by Eloísa Grifo: https://eloisagrifo.github.io/SymbolicPowers.pdf

10. **Quasi-regular sequences:** Introduce quasi-regular sequences and prove that a sequence is M-regular if and only if is is M-quasi regular in the local or graded cases. Essentially cover Theorems 16.2 and 16.3 of [Mat86].

    Reference: [Mat86, Section 6].

11. **Strongly stable ideals and Eliahou-Kervaire resolution:** Determine the minimal graded free resolution, Betti numbers and regularity of a strongly stable monomial ideal. Give an example and at least a sketch of the proof.

    Reference: [Pee10, Section 28].

12. **Stanley-Reisner rings:** Simplicial complexes. The Stanley-Reisner ring associated to a simplicial complex. The f-vector of a simplicial complex and its relation with the Hilbert function of the associated Stanley-Reisner ring. Shellable simplicial complexes and Cohen-Macauleyness.

    Reference: [BH98, Section 5.1].

13. **F-singularities:** The Frobenius map in rings of prime characteristic p and the singularities associated to it. Possible topics to cover here: Kunz's theorem, tight closure and weakly F-regular rings, F-pure rings, strongly F-regular rings and Fedder's criterion.

    Reference: these notes by De Stefani: https://a-destefani.github.io/ads/files/Notes.pdf

    There is enough here for more than one presentation.

14. **The Strong Lefschetz property for monomial complete intersections:** Talk about the weak and strong lefschetz properties of Artinian Gorenstein algebras and give the Reid-Roberts-Roitman proof that monomial complete intersections have the strong Lefschetz property.

    Reference: [RRR91] (essentialy Theorem 5) and [Har+13] (Sections 3.1 and 3.2).

15. **Other:** Feel free to also propose a topic that you want to learn and that we didn't cover in the course.

References

[AM69] M. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Company, 1969.

[BH98] W. Bruns and J. Herzog. *Cohen-Macaulay Rings*. Vol. 39. Cambridge University Press, 1998.

[BS13] M. P. Brodmann and R. Y. Sharp. *Local Cohomology: An Algebraic Introduction with Geometric Applications*. 2nd ed. Vol. 136. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2013.

[Eis13] D. Eisenbud. *Commutative Algebra: with a view toward algebraic geometry*. Vol. 150 of *Graduate texts in Mathematics*. Springer Science & Business Media, 2013.

[Har+13] T. Harima et al. *The Lefschetz properties*. Springer, 2013.

[HH11] J. Herzog and T. Hibi. *Monomial ideals*. Springer, 2011.

[HS06] C. Huneke and I. Swanson. *Integral closure of ideals, rings, and modules*. Vol. 336. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2006.

[Mat86] H. Matsumura. *Commutative ring theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1986.

[Pee10] I. Peeva. *Graded syzygies*. Vol. 14. Springer Science & Business Media, 2010.

[RRR91] L. Reid, L. G. Roberts, and M. Roitman. "On Complete Intersections and Their Hilbert Functions". In: *Canadian Mathematical Bulletin* 34.4 (1991), pp. 525–535.