Commutative Algebra: an introductory course

Problem set 5

Submit by 30/11

You only need to submit solutions for any 6 of the following problems:

1. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and M a finitely generated R-module. If $\operatorname{Tor}_1^R(k, M) = 0$, show that M is free.

2. Let $R = k[x, y, z]$, $M = R/(x, y)$ and $N = R/(x, z)$. Find a free resolution of N as an R-module and use it to compute $\operatorname{Tor}_i^R(M, N)$ for every $i \geq 0$.

3. Let I and J be ideals of a ring R. Show that $\operatorname{Tor}_1^R(R/I, R/J) \cong (I \cap J)/IJ$.

4. Let R be a Noetherian ring and N, M be R-modules with M finitely generated. If $\operatorname{Ext}_R^i(R/P, N) = 0$ for every $P \in \operatorname{Spec}(R)$, show that $\operatorname{Ext}_R^i(M, N) = 0$.

5. Let R be a ring and S a multiplicatively closed subset of R. For any R-modules M and N show that $S^{-1} \operatorname{Tor}_i^R(M, N) \cong \operatorname{Tor}_i^{S^{-1}R}(S^{-1}M, S^{-1}N)$ holds for all $i \geq 0$. (*Optional: if R is Noetherian and M is finitely generated then we also have* $S^{-1} \operatorname{Ext}_R^i(M, N) \cong \operatorname{Ext}_{S^{-1}R}^i(S^{-1}M, S^{-1}N)$).

6. Let R be an $\mathbb{N}$-graded Noetherian ring with R_0 a field and let M be a finitely generated $\mathbb{N}$-graded R-module. Prove the following identity for determining the Hilbert series of M in terms of the Hilbert series of R and the graded Betti numbers $b_{i,j} = \beta_{i,j}^R(M)$ of M:

$$\operatorname{HS}_M(t) = \operatorname{HS}_R(t) \left(\sum_{p \geq 0} \sum_{i \geq 0} (-1)^i b_{i,p} t^p \right).$$

7. Given any $n \geq 1$, give an example of a Noetherian local ring $(R, \mathfrak{m})$ of dimension n and depth zero.

8. Let R be the polynomial ring $k[x_1, \ldots, x_n]$, where k is a field. Suppose $f_1, \ldots, f_r$ is a homogeneous regular sequence in R of polynomials of degrees $d_1, \ldots, d_r \geq 1$. Show that the multiplicity of $R/(f_1, \ldots, f_r)$ is $e(R/(f_1, \ldots, f_r)) = d_1 \cdots d_r$.

9. Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence of R-modules. Suppose $x_1, \ldots, x_n$ is a sequence which is both A-regular and C-regular. Show that $x_1, \ldots, x_n$ is also B-regular.

10. Let R be a Noetherian ring, M a finitely generated R-module and I an ideal of R. Show that $\operatorname{depth}_I(M) \geq 2$ if and only if the natural homomorphism $M \longrightarrow \operatorname{Hom}_R(I, M)$ is an isomorphism.

11. Let R be a ring, M an R-module and $\underline{x} = x_1, \ldots, x_n$ a sequence of elements in R.

    (a) If $\underline{x}$ is M-regular, show that $\operatorname{Tor}_1^R(M, R/(\underline{x})) = 0$.

    (b) If in addition $\underline{x}$ is also R-regular, show that $\operatorname{Tor}_i^R(M, R/(\underline{x})) = 0$ for all $i \geq 1$.

Please be cool: include at least one joke in your solutions!