FINAL EXAM

Instructions: Justify your answers. You may use results from the homework sets, but make sure to carefully state such results. No calculators and no notes allowed. You may e.g., use part of Problem 4 to do part of Problem 1, even if you are unsuccessful with that part of Problem 4. You may use part (a) of a problem to do part (b) even if you have not solved (a), and so on.

Grading: This exam is worth 30 points. If you completed homework assignments, your homework bonus (out of 3 points) will be added to your score. You need a score of 12.5/30 or higher to pass this exam. More precisely, the following scale will be used:

A: $[26.5, 30]$, B: $[23, 26.5)$, C: $[19.5, 23)$, D: $[16, 19.5)$, E: $[12.5, 16)$, F: $[0, 12.5)$.

Problem 1. *Let S_4 be the symmetric group on $\{1, 2, 3, 4\}$.*

(a) (2 points) Show that the quaternion group of order 8 is not a subgroup of S_4.
(b) (2 points) Let H be a subgroup of S_4 which contains a 3-cycle and a 4-cycle. Show that $H = S_4$.
(c) (2 points) Let V be the Klein 4 subgroup of S_4 given by $V := \{e, (12)(34), (14)(23), (13)(24)\}$. Show that V is normal in S_4.
(d) (2 points) Show that $S_4/V \cong S_3$.

Problem 2. *Let G be a group. A subgroup K of G is said to be characteristic in G if K is stable under all automorphisms of G i.e., $\varphi(K) = K$ for all $\varphi \in \mathrm{Aut}(G)$.*

(a) (1 points) Show that a characteristic subgroup of G is normal in G.
(b) (2 points) Assume N is a normal subgroup of G and K is a characteristic subgroup of N. Show that K is normal in G.
(c) (2 points) Let p be a prime. Assume G is finite and that P is a p-Sylow subgroup of G. Show that the normalizer $N_G(P)$ of P in G is self-normalizing i.e. $N_G(N_G(P)) = N_G(P)$.
(d) (2 point) Assume G is finite and that P, Q are two distinct p-Sylow subgroups of G of order p^2. If $P \cap Q \neq \{e\}$, show that $N_G(P \cap Q)$ contains at least $p + 1$ Sylow p-subgroups of G.

Problem 3. *Let R be a commutative ring with 1.*

(a) (1 point) Assume R is an integral domain. Show that the polynomial $x^2 - 1 \in R[x]$ has at most two distinct roots in R.
(b) (1 point) Give an example of an integral domain R where $x^2 - 1 \in R[x]$ has just one distinct root in R.
(c) (1 point) Let p be a prime and $k \geq 2$ an integer. Show that $\mathbf{Z}/p^k\mathbf{Z}$ contains nonzero nilpotent elements and a unique maximal ideal.
(d) (1 point) Give an example where R has a unique maximal ideal and where $x^2 - 1 \in R[x]$ has strictly more than 2 distinct roots in R.
(e) (1 point) Give an example where R has no nonzero nilpotent elements and where $x^2 - 1 \in R[x]$ has strictly more than 2 distinct roots in R.

Problem 4. *Let p be an odd prime and let $n \geq 1$ be an integer.*

(a) (2 points) Show that a group of order $2p$ which is not abelian is isomorphic to the dihedral group D_{2p} of order $2p$.
(b) (1 points) Show that any group of order $2p$ is either dihedral or cyclic.
(c) (2 points) Let G be a group of order $2n$ with an element b of order n and an element a of order 2 which is not a power of b. Show that every element of G can be written uniquely in the form $a^i b^j$ with $0 \leq i \leq 1$ and $0 \leq j \leq n - 1$.
(d) (2 points) Give an example of a group of order 16 which is neither abelian nor dihedral but which is an example of (c) i.e., G has an element b of order 8 and an element a of order 2 which is not a power of b. Justify your answer.

Problem 5.

(a) (1 point) Show that $x^3 + 6x + 10$ is irreducible in $\mathbf{Q}[x]$ but reducible in $\mathbf{R}[x]$.
(b) (1 point) Show that $x^4 + 1$ is irreducible in $\mathbf{Q}[x]$.
(c) (1 point) Show that $x^4 + 1$ is reducible in $\mathbf{F}_3[x]$.

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