FINAL EXAM

Instructions: Justify your answers. You may use results from the homework sets, but make sure to carefully state such results. No calculators and no notes allowed. You may e.g., use part of Problem 4 to do part of Problem 1, even if you are unsuccessful with that part of Problem 4. You may use part (a) of a problem to do part (b) even if you have not solved (a), and so on.

Grading: This exam is worth 30 points. If you completed homework assignments, your homework bonus (out of 3 points) will be added to your score. You need a score of 12.5/30 or higher to pass this exam. More precisely, the following scale will be used:

A: $[26.5, 30]$, B: $[23, 26.5)$, C: $[19.5, 23)$, D: $[16, 19.5)$, E: $[12.5, 16)$, F: $[0, 12.5)$.

Problem 1 (6 points). *Let $n \geq 2$ be an integer, let S_n be the symmetric group on $\{1, 2, \dots, n\}$ and let $\tau \in S_n$ be a transposition.*

(a) *(1 point) Compute the size of the conjugacy class of τ in S_n.*
(b) *(3 points) Describe an isomorphism between the centralizer $\mathrm{Cent}_{S_n}(\tau)$ of τ and the direct product $\mathbf{Z}/2 \times S_{n-2}$.*
(c) *(1 points) For $n > 2$ show that there is no n-cycle in S_n which commutes with τ.*
(d) *(1 point) Give an example of $n > 2$, a transposition τ and an element of order n in S_n which commutes with τ.*

Problem 2 (5 points). *Assume N and M are two subgroups of a group G.*

(a) *(2 points) Suppose N is normal in G. Show that NM is a subgroup of G.*
(b) *(1 points) Now assume that both N and M are normal in G. Show that NM is normal in G.*
(c) *(2 points) Finally, suppose that both N and M are normal in G, that $NM = G$ and that $N \cap M = \{e\}$. Show that $G \cong N \times M$.*

Problem 3 (9 points). *Let G be a finite group.*

(a) *(3 points) Assume $|G| = 33$. Show G is cyclic.*
(b) *(3 points) Assume $|G| = 5^3 \cdot 31$. Show that G has a nontrivial, normal p-Sylow subgroup for some prime p.*
(c) *(3 points) Assume $|G| = 165$ and that G has a normal 5-Sylow subgroup. Show that G is abelian.*

Problem 4 (3 points). *Let $\varphi : R \to S$ be a homomorphism between commutative rings with 1.*

(a) *(1 point) Show that the inverse image of a prime ideal in S is a prime ideal in R.*
(b) *(1 point) Show that every maximal ideal of S is a prime ideal of S.*
(c) *(1 point) Show by example that the inverse image of a maximal ideal in S need not be maximal in R.*

Problem 5 (7 points). *Let $f(x) \in \mathbf{Z}[x]$.*

(a) *(1 point) Assume $f(x) = x^3 - 2x + 14$. Show that $f(x)$ is irreducible in $\mathbf{Q}[x]$ but reducible in $\mathbf{R}[x]$.*
(b) *(1 point) Assume that f is monic and that for some prime p, the reduction of f modulo p is irreducible in $\mathbf{F}_p[x]$. Show that $f(x)$ is irreducible in $\mathbf{Z}[x]$.*
(c) *(1 point) Assume f is irreducible in $\mathbf{Q}[x]$. Is f necessarily irreducible in $\mathbf{Z}[x]$? Explain.*
(d) *(1 point) Assume $f(x) = x^4 + ax^2 + b$ for some $a, b \in \mathbf{Z}$. Show that the roots of f in $\mathbf{C}$ have the form $\pm\alpha, \pm\beta$ for some $\alpha, \beta \in \mathbf{C}$ and that $(\alpha\beta)^2 \in \mathbf{Z}$.*
(e) *(2 points) Let $f(x)$ be as in (d). Show that f is irreducible over $\mathbf{Q}$ if and only if none of $\alpha^2, \alpha - \beta, \alpha + \beta$ lie in $\mathbf{Q}$.* Hint: *Compute also $\alpha^2 + \beta^2$.*
(f) *(1 point) As a concrete example of (d)-(e), factor $f(x) = x^4 + 4$ into irreducibles over $\mathbf{Q}$.*

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