MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Sven Raum

Tentamensskrivning i
Foundations of Analysis
7.5 hp
22nd August 2019

Please read carefully the general instructions:

- During the exam any textbook, class notes, or any other supporting material is forbidden.
- In particular, calculators are not allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- The exam has 5 questions. If you handed in homework, you solve problem 5 and select 3 problems among questions 1-4 which you solve. State clearly which problems you chose. If you opted out of the homework, you solve problems 1-5.
- Each question is graded on a 0-20 scale.
- A score of at least 50 points will ensure a pass grade.
- The exam is returned on 2nd September at 10 o'clock in sal 16, hus 5.

GOOD LUCK!

1. Sets

   (a) Define countability and prove that the set of natural numbers $\mathbb{N}$ is countable.

   (b) Provide an example of an uncountable set. Verify that it is indeed uncountable.

   (c) Let X be a set. Show that the following two statement are equivalent:

   - X is finite or countable.
   - There is an injective map $X \to \mathbb{N}$.

   (d) Let X be a countable set and $X \to Y$ be a surjective map onto another set Y. Show that Y is countable or finite. (Even if you heard of it, you are not allowed to use the axiom of choice.)

   Idea for a solution.

   (a) A set X is countable if there is a bijection $X \to \mathbb{N}$. The natural numbers are countable, since the identity map $\mathbb{N} \to \mathbb{N}$ is a bijection.

   (b) The set of all binary sequences $\{0,1\}^{\mathbb{N}}$ is uncountable. Indeed let $\varphi : \mathbb{N} \to \{0,1\}^{\mathbb{N}}$ be any map. Define a binary sequence f by

$$f(k) = \begin{cases} 0 & \text{if } \varphi(k)_k = 1 \\ 1 & \text{if } \varphi(k)_k = 0 \end{cases}$$

   Then f is different from any element in the image of φ, so the φ cannot be surjective. In particular there is not bijection $\mathbb{N} \to \{0,1\}^{\mathbb{N}}$.

   (c) By definition X is finite if there is some $n \in \mathbb{N}$ and a bijection $X \to \{0, \ldots, n-1\}$. (Remark: here is a little sublety about the empty set being finite!) Since the latter set maps injectively into $\mathbb{N}$, every finite set X admits an injection into $\mathbb{N}$. So does every countable set, as inspecting the definition shows. Vice versa assume that X is a set and $\varphi : X \to \mathbb{N}$ is an injection. Define a injection by

$$\psi : X \to \mathbb{N} : \psi(x) = |\{y \in X \mid \varphi(y) < x\}|$$

   Then either ψ is surjective (iff X is infinite) or $\psi(X) = \{0, \ldots, n-1\}$ for some $n \in \mathbb{N}$ that hence satisfies $n = |X|$.

   (d) Since X is countable, we may from in the beginning assume that $X = \mathbb{N}$. Let $\varphi : \mathbb{N} \to Y$ be a surjection. For every $y \in Y$ set $\psi(y) = \min\{x \in X \mid \varphi(x) = y\}$. Note that this is well-defined, since φ is surjective and every subset of $\mathbb{N}$ has a unique smallest element. By construction ψ is injective, so positive solution to the previous question implies that Y is countable or finite.

2. Metric spaces and topology

   (a) Let (X, d) be a metric space.

      i. Define the notion of an open subset of X.

      ii. Define the notion of a closed subset of X.

   (b) Let $n \in \mathbb{N}_{\geq 1}$ and consider the function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d(x, y) = \sum_{i=1}^{n} |x_i - y_i|.$$

      i. Show that $(\mathbb{R}^n, d)$ is a metric space.

      ii. Show that $(0,1)^n \subset \mathbb{R}^n$ is open with respect to the metric d.

   (c) Provide an example of a X metric space which contains a non-empty compact open subset S such that $S \neq X$.

   (d) Show that $[0,1] \subset \mathbb{R}$ is compact, without referring to the Heine-Borel theorem.

Idea of solutions.

(a) A subset $O \subset X$ is open if for every $x \in O$ there is some $\varepsilon > 0$ such that $B(x, \varepsilon) \subset O$. A subset $A \subset X$ is closed if for every convergent sequence $(a_n)_n$ of elements in A, the limit point satisfies $\lim_{n\to\infty} a_n \in A$.

(b) On order to verify that $(\mathbb{R}^n, d)$ is a metric space, one has to show that:

- For all $x, y \in \mathbb{R}^n$ we have $d(x, y) \geq 0$: this is clear from the definition.
- If $d(x, y) = 0$, then $x = y$: indeed if $d(x, y) = 0$, then for all $i \in \{1, \ldots, n\}$ we have $|x_i - y_i| = 0$ and hence $x_i = y_i$.
- For all $x, y \in \mathbb{R}^n$ we have $d(x, y) = d(y, x)$: this follows from $|x_i - y_i| = |y_i - x_i|$.
- The triangle inequality: this follows from the triangle inequality for the absolute value $|x_i - y_i|$.

In order to show that $(0, 1)^n$ is open with respect to d, the quickest way is to notice that, denoting $x = (\frac{1}{2}, \ldots, \frac{1}{2})^t \in \mathbb{R}^n$, we have $(0, 1)^n = B(x, 1/2)$. In every metric space balls are open, which is very quick to show with the triangle inequality.

(c) The discrete metric space with two points is such an example.

(d) We have to show that every open cover of $[0, 1]$ has a finite subcover. Let $(U_i)_{i\in I}$ be an open cover of $[0, 1]$. Define

$$\mathcal{S} = \{x \in [0, 1] \mid \exists \text{ finite subset } J \subset I : [0, x] \subset \bigcup_{i\in J} U_i\}.$$

Note that $\mathcal{S}$ is bounded and non-empty (since $0 \in \mathcal{S}$). By completeness of $\mathbb{R}$ there is a minimal upper bound of $\mathcal{S}$, which we denote by x_0. Let $i_0 \in I$ be some index such that $x_0 \in U_{i_0}$ and let $\varepsilon > 0$ be chosen such that $B(x, 2\varepsilon) \subset U_{i_0}$, where we take balls in the metric space $[0, 1]$. By definition of $\mathcal{S}$ there is some finite subset $J \subset I$ such that $[0, x_0 - \varepsilon] \subset \bigcup_{i\in J} U_i$. So, denoting $x_1 = \min\{x_0 + \varepsilon, 1\}$ the family $(U_i)_{i\in J\cup\{i_0\}}$ covers $[0, x_1]$. Since x_0 is an upper bound for $\mathcal{S}$, this shows $x_0 = 1$.

3. Differentiation

(a) State the implicit function theorem.

(b) Let $O \subset \mathbb{R}^n$ be an open subset and $f : O \to \mathbb{R}$ be differentiable. Show that f is continuous.

(c) i. Prove Rolle's theorem.

ii. Show that the converse of Rolle's theorem does not hold, by providing an example of a differentiable function $f : [-1, 1] \to \mathbb{R}$ such that $f'(0) = 0$ holds, but f does not have a local extremum at 0.

(d) For $k \in \mathbb{N}$ let $f_k : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function satisfying $f_k(0) = 0$. Assume that for all $k \in \mathbb{N}$ and all $x \in \mathbb{R}^n$ we have $\|f'_k(x)\| \leq \frac{1}{k^2}$.

i. Show that the series $f(x) = \sum_{k=0}^{\infty} f_k(x)$ converges absolutely for all $x \in \mathbb{R}^n$.

ii. Show that f is continuously differentiable.

Idea of solutions.

(a) The theorem can be found for example as Theorem 9.28 in Rudin's book.

(b) Let $f : O \to \mathbb{R}$ be as in the statement. It suffices to show that f is continuous on every open square inside O and we can thus restrict to the case $n = 1$. For $x \in O$ we us the definition of differentiability at x and calculate

$$|f(x) - f(x + h)| = h \left| \frac{f(x + h) - f(x)}{h} \right| \longrightarrow 0 \cdot f'(x) = 0$$

So f is continuous at x.

(c) i. Rolle's theorem states that for every differentiable function on a non-trivial compact interval $f : [a, b] \to \mathbb{R}$ satisfying $f(a) = f(b)$ there is some $a < x < b$ such that $f'(x) = 0$. For its proof we observe that either f is constant (in which case the conclusion is trivial) or f attains a local extremum at some point $x \in (a, b)$. Without loss of generality, we may assume that x is a local minimum. Then agreement of left and right derivative imply that

$$f'(x) = \lim_{h\to 0, h>0} \frac{f(x+h) - f(x)}{h} \geq 0$$
$$f'(x) = \lim_{h\to 0, h<0} \frac{f(x+h) - f(x)}{h} \leq 0.$$

It follows that indeed $f'(x) = 0$ holds.

ii. The function $f(x) = x^3$ does the job.

(d) Since $|f_k'(x)| \leq \frac{1}{k^2}$ we also fine $|f(x)| \leq \frac{1}{k^2}\|x\|$ for all $x \in \mathbb{R}^n$. Hence we obtain absolute converges thanks to the estimate

$$\sum_k |f_k(x)| \leq \|x\| \sum_k \frac{1}{k^2} < \infty.$$

By a direct estimate, also $\sum_k f_k'(x)$ converges (absolutely). One can now apply Theorem 7.17 of Rudin to conclude that $\sum_k f_k$ is differentiable.

4. Integration

(a) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Define the Riemann integral of f.

(b) Show that the function

$$\mathbb{1}_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$$
$$\mathbb{1}_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

is not Riemann integrable on any non-trivial compact interval.

(c) Find a monotone increasing function $\alpha : [0, 1] \to \mathbb{R}$ such that for all continuous functions $f : [0, 1] \to \mathbb{R}$ the following equality holds:

$$\int_0^1 f \mathrm{d}\alpha = \frac{1}{2}\left(f(0) + f(1)\right).$$

Remember to prove all claims you make.

(d) For a Riemann integrable function $f : [a, b] \to \mathbb{R}$ define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\mathrm{d}t.$$

i. Show that for any choice of f as above, the function F is continuous.

ii. Provide an example of a function f as above, such that F is not differentiable.

Idea of solutions.

i. Let $f : [0, 1] \to \mathbb{R}$ be continuous. The Riemann integral of f can be defined by first defining upper and lower sums over partitions of $[0, 1]$ and then stating that their limits (the upper and lower integral) agree for continuous functions and their common value is the Riemann integral.

ii. One uses density of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$ in order to show that the upper integral of $\mathbb{1}_{\mathbb{Q}}$ equals $|b - a|$ and its lower integral equals zero.

iii. The function

$$\alpha(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{2} & \text{if } 0 < x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

does the job. This is proven by calculating the upper and lower sums for the Riemann-Stieltjes integral explicitly and making use of continuity of f at the points 0 and 1.

iv. A. Riemann integrable functions are bounded, say $|f| \leq M$. Then one estimates for $x < y$

$$|F(x) - F(y)| = |\int_x^y f(t)\mathrm{d}t| \leq \int_x^y |f(t)|\mathrm{d}t \leq |y - x|b.$$

It follows that F is even uniformly continuous.

B. Taking $f = \mathbb{1}_{[0,1]}$ on the interval $[-1, 1]$ a short calculation shows that $F(x) = x\mathbb{1}_{[0,1]}(x)$, which is not differentiable.

5. True / false questions

Please indicate your answers on the separate answer sheet.

(a) If $A \subset \mathbb{R}$ is countable and $B \subset \mathbb{R}$ is an arbitrary set, then $A \cap B$ is countable.

(b) If a subset E of the real numbers is bounded above, and $x = \sup E$, then $x \in E$.

(c) The set $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}_{\geq 1}\} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$ is compact.

(d) Let X be a metric space and $A \subset X$ a proper subset with some isolated point. Then A is not open.

(e) Let X be a metric space. If $E \subset X$ a perfect subset and $F \subset E$ a subset that is closed in X. Then F is perfect.

(f) Let $(p_n)_{n\in\mathbb{N}}$ be a sequence in a metric space X. Then $(p_n)_n$ converges to $p \in X$ if and only if every neighbourhood of p contains all but finitely many members of $(p_n)_n$.

(g) Given any real sequence $(a_n)_{n\in\mathbb{N}}$, we have $\sum_{n=0}^{\infty}(a_n - a_{n+1}) = a_0$.

(h) The function $f : (1, +\infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is uniformly continuous.

(i) If $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, then f has at most countably many points of discontinuity.

(j) If $f : X \to Y$ is a continuous function between compact metric spaces and $E \subset X$ is closed, then $f(E) \subset Y$ is a closed subset.

(k) Put $X = (-\infty, 0) \cup (0, \infty)$ and let $f : X \to \mathbb{R}$ be a differentiable function satisfying $f'(x) = 0$ for all $x \in X$. Then f is constant.

(l) Let $f : \mathbb{R} \to \mathbb{R}$ be a function whose derivatives of all orders exist and are continuous. Then $f(x) = \sum_{n\in\mathbb{N}} \frac{f^{(n)}(0)}{n!}x^n$ for every real number for which the right-hand side converges.

(m) Let $f : [0, 1] \to \mathbb{R}$ be a function whose absolute value $|f|$ is Riemann integrable. Then also f is Riemann integrable.

(n) Let $\alpha, \beta : [0, 1] \to \mathbb{R}$ be two monotone increasing functions and assume that $\mathcal{R}(\alpha) = \mathcal{R}(\beta)$. Further assume that for all $f : [0, 1] \to \mathbb{R}$ in $\mathcal{R}(\alpha)$ equality

$$\int_0^1 f\mathrm{d}\alpha = \int_0^1 f\mathrm{d}\beta$$

holds. Then $\alpha = \beta$.

(o) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $\mathrm{C}'([0, 1], \mathbb{R})$ which converges uniformly to $f \in \mathrm{C}([0, 1], \mathbb{R})$. Then f is also continuously differentiable.

(p) Let X be a metric space and $E \subset X$ dense subset. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of real-valued continuous functions X such that the sequence of restrictions $(f_n|_E)_n$ converges uniformly. Then also $(f_n)_n$ converges uniformly.

(q) Let $K \subset \mathrm{C}([0,1], \mathbb{R})$ be a compact set. Then there is $B \in \mathbb{R}_{\geq 0}$ such that for all $f \in K$ and for all $x \in [0,1]$ we have $|f(x)| \leq B$.

(r) Assume that $f : \mathbb{R}^n \to \mathbb{R}$ admits all partial derivatives $\frac{\partial f}{\partial x_i}$, $i \in \{1, \dots, n\}$. Then f is differentiable.

(s) Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Then there is some element $v \in \mathbb{R}^n \setminus \{0\}$ such that $\frac{\partial f}{\partial x_n} = D_v f$.

(t) Denote by $\mathrm{d} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ the Euclidean distance function on $\mathbb{R}^2$. Then inverse function theorem applies at every point in $\mathbb{R}^2$ to the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x) = \mathrm{d}(0, x)$.

ANSWER SHEET FOR QUESTION 5

EXAM CODE: ....

- Answer Question 5 by placing a clear mark (either ✓or ×) in the box that corresponds to your answer of the table below. Please, mark your answer ONCE only.
- In order to provide a fair exam, that does not encourage random answering, the following system is applied to calculate your points on question 5. For each correct answer you will be awarded 1 point. For each incorrect answer you will lose 1 point. If the number of incorrect answers is higher than the number of correct answers, then the total mark awarded for this question be 0.

	True	False
a.		x
b.		x
c.	x	
d.		x
e.		x
f.	x	
g.		x
h.	x	
i.	x	
j.	x	
k.		x
l.		x
m.		x
n.		x
o.		x
p.	x	
q.	x	
r.		x
s.	x	
t.		x