MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Sven Raum

Tentamensskrivning i
Foundations of Analysis
7.5 hp
2nd October 2019

Please read carefully the general instructions:

- During the exam any textbook, class notes, or any other supporting material is forbidden.
- In particular, calculators are not allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- The exam has 5 questions. If you handed in homework, you solve problem 5 and select 3 problems among questions 1-4 which you solve. State clearly which problems you chose. If you opted out of the homework, you solve problems 1-5.
- Each question is graded on a 0-20 scale.
- A score of at least 50 points will ensure a pass grade.
- The exam is returned on 15th October at 11 o'clock in office 402, house 6.

GOOD LUCK!

1. Ordered fields

   (a) Define the notion of an ordered field.

   (b) Show that the rational numbers form an ordered field.

   (c) Prove that the rational numbers do not have the least-upper-bound property.

   (d) Show that for an ordered field F the following statements are equivalent:

   - F has the least-upper-bound property.
   - Every non-empty subset of F that has a lower bound, also has a largest lower bound.
   - Every non-empty, bounded subset of F has a least upper bound as well as a largest lower bound.

   Idea of solution.

   (a) An ordered field is a field F with an order $<$ such that the following two axioms hold:

   - for all $x, y, z \in F$: $x < y \Longrightarrow x + z < y + z$
   - for all $x, y \in F$: $x > 0$ and $y > 0 \Longrightarrow xy > 0$

   (b) We have to check the two axioms. For the first one, we have to recall the order on $\mathbb{Q}$: for $a, c \in \mathbb{Z}$ and $b, d \in \mathbb{N}_{\geq 1}$ we have $\frac{a}{b} < \frac{c}{d}$ if and only if $ad < bc$. This immediately implies that in $\mathbb{Q}$ we have $x < y$ if and only if $0 < y - x$, which allows to conclude that the first axiom holds. Validity of the second axiom similarly follows from the fact that the product of two positive natural numbers is a positive natural number.

   (c) Let $A = \{x \in \mathbb{Q} \mid x^2 < 2\}$. Then 2 is an upper bound of A, so A is bounded from above. We show that it cannot have a least upper bound. Indeed, if u is any upper bounder of A, then one of the following three statements is true: either $u^2 < 2$, $u^2 = 2$ or $u^2 > 2$. If $u^2 < 2$ was the case, then for large $n \in \mathbb{N}$ we have

   $$(u + \frac{1}{n})^2 = u^2 + \frac{2}{n}u + \frac{1}{n^2} < 2$$

   which contradicts the assumption that u is an upper bound of A. Further, we can exclude the case $u^2 = 2$, since $\mathbb{Q}$ does not contain a root of 2. It follows that $u^2 > 2$ must hold. But, then for large $n \in \mathbb{N}$

   $$(u - \frac{1}{n})^2 = u^2 - \frac{2}{n}u + \frac{1}{n^2} > 2$$

   showing that there is a smaller upper bound of A than u.

   (d) The equivalence between the first two items is derived by using the additive inverse: u is an upper bound of A if and only if $-u$ is a lower bound for $-A$. Together the first two items imply the last one. It remains to conclude the first item from the last. If $A \subset F$ is bounded from above and non-empty, we take $a_0 \in A$ and consider the bounded set $B = \{a \in A \mid a \geq a_0\}$. The least upper bound of B, which exists by the last item, is then also a least upper bound of A.

2. Uniform continuity

   (a) Define the notion of a uniformly continuous function between two metric spaces.

   (b) Denote by $\mathbb{R}_{\geq 0}$ the set of non-negative real numbers. For $x \in \mathbb{R}_{\geq 0}$ and $n \in \mathbb{N}_{\geq 1}$ denote by $x^{1/n}$ the unique element of $\mathbb{R}_{\geq 0}$ satisfying $(x^{1/n})^n = x$. Show that for all $n \in \mathbb{N}_{\geq 1}$ the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $f(x) = x^{1/n}$ is uniformly continuous.

   (c) Show that every continuous function on a compact metric space is uniformly continuous.

   (d) Let $S \subset X$ be a dense subset of a metric space, let Y be a complete metric space and let $f : S \to Y$ be a uniformly continuous function. Show that there is a unique continuous function $g : X \to Y$ satisfying $g(x) = f(x)$ for all $x \in S$.

Idea of solution.

(a) Let X, Y be metric spaces. A function $f : X \to Y$ is called uniformly continuous if for all $\varepsilon > 0$ there is some $\delta > 0$ such that for all $x_1, x_2 \in X$ that satisfy $d(x_1, x_2) < \delta$ we have $d(f(x_1), f(x_2)) < \varepsilon$.

(b) Let $\varepsilon > 0$ and write $\delta_0 = (\frac{\varepsilon}{4})^n$. Consider the restriction of f to the interval $[\delta_0, \infty)$. The derivative of f on this interval equals $x \mapsto \frac{1}{n} x^{-\frac{n-1}{n}}$, which is bounded by some $C > 0$. Let $\delta = \min\{\frac{\varepsilon}{2C}, \delta_0\}$. We can show uniform continuity separately on the intervals $[0, \delta_0]$ and $[\delta_0, \infty)$. For the first case let $x, y \in [0, \delta_0)$. Then

$$d(x^{1/n}, y^{1/n}) \leq x^{1/n} + y^{1/n} \leq \frac{\varepsilon}{2} < \varepsilon.$$

If $x, y \in [\delta_0, \infty)$ satisfy $d(x, y) < \delta$, then the choice of C implies that $d(x^{1/n}, y^{1/n}) \leq C d(x, y) < \varepsilon$.

(c) This is proven in the course book – see Theorem 4.19.

(d) Let $x \in X$ and let $(x_n)_n$ and $(y_n)_n$ be two sequences of elements in S such that $x_n \to x$. Since f is uniformly continuous, the sequences $(f(x_n))_n$ and $f((y_n))_n$ are Cauchy in Y and thus by completeness converge to points say a and b in Y, respectively. We claim that $a = b$. Indeed, if $\varepsilon > 0$ there is some $\delta > 0$ such that whenever $w, z \in X$ satisfy $d(w, z) < \delta$, then $d(f(w), f(z)) < \varepsilon$. Let $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $d(x_n, x) < \delta/2$ and $d(y_n, x) < \delta/2$. Then a short calculation shows that $d(f(x_n), f(y_n)) < \varepsilon$ for all $n \geq n_0$. This implies that $d(a, b) \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we showed that $a = b$.

We can now define a function $g : X \to Y$ by declaring $g(x)$ to be the unique limit of a sequence $(f(x_n))_n$ where $(x_n)_n$ is any sequence of elements in S converging to x. This makes sense by the first paragraph. By construction g is continuous and extends f.

3. Differentiation

(a) Let $O \subset \mathbb{R}^n$ be an open subset. Define the notion of differentiable function $f : O \to \mathbb{R}^m$.

(b) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function whose partial derivatives exist and are continuous. Show that f is differentiable.

(c) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuously differentiable. Show that there is some constant $M > 0$ such that for all $x, y \in \mathbb{R}^n$ satisfying $\|x\|, \|y\| \leq 1$ the following inequality holds.

$$\|f(x) - f(y)\| \leq M\|x - y\|.$$

Idea of solutions.

(a) This is Definition 9.11 of the course book.

(b) This is the second part of the proof of Theorem 9.21

(c) This is a special case of Theorem 9.19

4. Integration

(a) Give an example of a sequence of continuous functions $f_n : [0, 1] \to \mathbb{R}$ such that $\int_0^1 f_n(x)dx = 1$ for all $n \in \mathbb{N}$, but $\lim_{n\to\infty} f_n = 0$ pointwise.

(b) For $n \in \mathbb{N}$, let $f_n : [0, 1] \to \mathbb{R}$ be some continuous functions such that $\lim_{n\to\infty} f_n = 0$ pointwise. Assume that $f_{n+1}(x) \leq f_n(x)$ holds for all $n \in \mathbb{N}$ and all $x \in [0, 1]$. Show that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$

(c) Assume that f is a Riemann integrable function on $[0, 1]$. Show that f^2 is Riemann integrable.

Idea of solutions.

(a) One can choose the following sequence

$$f_n(x) = \begin{cases} n^2 x & x \in [0, \frac{1}{n}] \\ 2n - n^2 x & x \in [\frac{1}{n}, \frac{2}{n}] \\ 0 & \text{otherwise.} \end{cases}$$

(b) This follows from a combination of Theorem 7.13 and 7.16.

(c) The proof of Theorem 6.11 can be applied.

5. True / false questions

Please indicate your answers on the separate answer sheet.

(a) Every finite subset of $\mathbb{Q}$ has a supremum.

(b) If F is an ordered field and $x \in F$, then $x \le x^2$.

(c) The union of any collection of countable sets is countable.

(d) The subset $\mathbb{Q} \subset \mathbb{R}$ is neither open nor closed.

(e) If $A \subset \mathbb{R}$ is a bounded subset, then $\sup A$ is a limit point of A.

(f) If K is a compact metric space and $(U_i)_{i\in I}$ is an open cover of K, such that for every point $x \in K$ there are at least two sets in $(U_i)_i$ that contain x, then there is a finite subcover of $(U_i)_i$ having the same property.

(g) Every convergent sequence in a metric space is a Cauchy sequence.

(h) If $\sum_n a_n = A$ and $\sum_n b_n = B$ are two absolutely convergent series of real numbers, then $\sum_n a_n b_n = AB$.

(i) The function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is uniformly continuous.

(j) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $x \mapsto |f(x)|$ is a continuous function. Then f is continuous.

(k) If $f, g : [0, 1] \to \mathbb{R}$ are two functions such that f and $f + g$ are differentiable, then also g is differentiable.

(l) If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function and $x \in \mathbb{R}$ is such that $f'(x) > 0$, then there is a neighbourhood of x on which f is monotone increasing.

(m) Every differentiable function on $[0, 1]$ is a uniform limit of polynomials

(n) There are two bounded functions $f, g : [0, 1] \to \mathbb{R}$ such that

$$\underline{\int_0^1} (f + g)(x)dx > \underline{\int_0^1} f(x)dx + \underline{\int_0^1} g(x)dx.$$

(o) Every monotonically increasing function on $[0, 1]$ is Riemann-Stieltjes integrable with respect to any monotonically increasing function α.

(p) The sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \frac{x}{n}$ converges uniformly to the zero function.

(q) Let E be a metric space and $(f_n)_n$ a sequence of continuous functions on E converging pointwise to some function f. Then $f_n \to f$ uniformly if and only if f is continuous.

(r) If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $x \in \mathbb{R}^n$, then all directional derivatives of f at x exist.

(s) There is a sequence of Riemann integrable functions on $[0, 1]$ which uniformly converge to the indicator function

$$\mathbb{1}_{\mathbb{Q}\cap[0,1]}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

(t) The Inverse Function Theorem applies at every point to the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = e^{xy}$.

ANSWER SHEET FOR QUESTION 5

EXAM CODE: ....

- Answer Question 5 by placing a clear mark (either ✓or ×) in the box that corresponds to your answer of the table below. Please, mark your answer ONCE only.
- In order to provide a fair exam, that does not encourage random answering, the following system is applied to calculate your points on question 5. For each correct answer you will be awarded 1 point. For each incorrect answer you will lose 1 point. If the number of incorrect answers is higher than the number of correct answers, then the total mark awarded for this question be 0.

	True	False
a.	x	
b.		x
c.		x
d.	x	
e.		x
f.	x	
g.	x	
h.		x
i.		x
j.		x
k.	x	
l.	x	
m.	x	
n.	x	
o.		x
p.		x
q.		x
r.	x	
s.		x
t.		x