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Written Exam
Logic II
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The maximum score on this written exam is 40 points (p). Grading (after inclusion of bonus points): A requires at least 32 p, B at least 28 p, C at least 22 p, D at least 18 p and E at least 16 p. Maximum score for each problem is indicated below.

The allowed time for the exam is five hours. No aids are permitted except paper and pen. Write clearly and motivate your answers carefully.

1. Prove that the following functions are primitive recursive.

   (a) $\mathrm{pd} : \mathbb{N} \to \mathbb{N}$ given by $\mathrm{pd}(0) = 0$ and $\mathrm{pd}(x) = x - 1$ if $x > 0$.

   (b) $\dot{-} : \mathbb{N}^2 \to \mathbb{N}$ given by $x \dot{-} y = x - y$ if $x \geq y$, and $x \dot{-} y = 0$ if $x < y$. (Hint: use (a) and primitive recursion on y)

   (3p)

2. *(Un)decidable properties of partial recursive functions.* Determine (with proof) whether each of the following sets is recursive or not.

   (a) $\{(x, y) \in \mathbb{N}^2 : \phi^1_x = \phi^1_y\}$

   (b) $\{x \in \mathbb{N} : W_x \text{ is finite}\}$

   (c) $\{(x, y, n) \in \mathbb{N}^3 :$ Turing machine with index x stops on input y at the nth step $\}$

   Hint: Rice's theorem may be useful. (6p)

3. (a) When is a class of structures $\mathcal{A}$ in a language L axiomatisable?

   (b) Prove that the class of finite partial orders is not axiomatizable in the language $L_{\mathrm{ord}} = \{\leq\}$

   (c) Prove that the class of infinite partial orders *is* axiomatizable in L_{ord}.

   (5 p)

4. Let L be any language. Prove that for any infinite L-structure $\mathcal{M}$ there is a non-isomorphic L-structure $\mathcal{N}$, so that $\mathcal{M} \equiv \mathcal{N}$. (5 p)

5. State the three main equivalent forms of the axiom choice (AC, Zorn's lemma and the well-ordering principle). Pick a suitable one of them, and use it to prove that every partial ordering of a set can be extended to a linear ordering of the set, i.e. prove that for every set S, and $P \subseteq S^2$ a partial order on S, there is a linear order $L \subseteq S^2$ on S such that $L \supseteq P$.

(5 p)

6. Calculate the cardinalities (you may assume the axiom of choice) of the following sets and order them according to size :

   (a) $S_1 = \mathbb{R}^{\mathbb{Q}}$
   (b) $S_2 = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$ (Hint: such continuous functions are determined by their values for rational numbers.)
   (c) $S_3 = \{f : \mathbb{N} \to \mathbb{N} \mid f \text{ is increasing}\}$
   (d) $S_4 = \{f : \mathbb{N} \to \mathbb{N} \mid f \text{ is non-increasing}\}$

(5 p)

7. State Gödel's first incompleteness theorem. Prove that the theory

$$\mathrm{PA} \cup \{F_n : n > 2\}$$

is incomplete. Here PA is Peano Arithmetic and

$$F_n = \forall x \forall y \forall z((x+1)^n + (y+1)^n \neq (z+1)^n),$$

and t^n is short for $t \cdot \cdots \cdot t$ (n times). Fermat's Last Theorem (proved by Andrew Wiles 1995) states that $\mathbb{N} \models F_n$ for all $n > 2$. It is not known whether this theorem can be proved in PA. (5p)

8. A well-known theorem in number theory, due to Lagrange, states that every non-negative integer can be written as a sum of four squares of integers. E.g. $23 = 3^2 + 3^2 + 2^2 + 1^2$.

   (a) Use Lagrange's theorem to prove that the set of natural numbers is first-order definable in the ring of integers $\mathcal{Z} = (\mathbb{Z}; +, \cdot, 0, 1)$.
   (b) Prove that the ring of integers is undecidable, i.e. that $\mathrm{Th}(\mathcal{Z}) = \{\#F : \mathcal{Z} \models F \text{ and } F \text{ is a closed } L_{\mathrm{ring}}\text{-formula}\}$ is not recursive.

(6p)