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Written Exam
Logic II, MM7022
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Written Exam Logic II

The maximum score on this written exam is 40 points. Grading (after inclusion of bonus points): A requires at least 32/40, B at least 28/40, C at least 22/40, D at least 18/40 and E at least 16/40. The maximum score for each problem is indicated below.

The allowed time for the exam is five hours. No aids are permitted except paper and pen. Write clearly and justify all answers carefully. You may make use of any theorems from the course.

1. (5p) Which of the following situations can occur? In each case, either give an example, or show that no such elementary embedding can exist.

   (a) An elementary embedding $f : M \to N$, such that $|M| < |N|$.
   (b) An elementary embedding $f : M \to N$, such that $|M| > |N|$.
   (c) An elementary embedding $f : M \to N$, such that $|M| = |N|$, but f is not a bijection.

2. (6p) For each of the following sets, determine whether its cardinality is equal to $|\mathbb{R}|$, or strictly greater, or strictly less.

   (a) $\mathbb{R}^{<\omega}$, the set of all finite sequences of reals;
   (b) $\mathrm{Bij}(\mathbb{R}, \mathbb{R})$, the set of all bijections $\mathbb{R} \to \mathbb{R}$;
   (c) $\mathrm{Mon}(\mathbb{N}, \mathbb{N})$, the set of all monotone functions $(\mathbb{N}, \leq) \to (\mathbb{N}, \leq)$.

3. (6p) We write $\mathrm{Drv}(n, v)$ for the formula in $\mathcal{L}_{PA}$ representing the set

   $$\{(\#\varphi, \#D) \mid \varphi \text{ a closed formula, } D \text{ a derivation of } \varphi \text{ in PA}\} \subseteq \mathbb{N}^2,$$

   and $\mathrm{Con}(PA)$ for the closed formula $\neg\exists v\ \mathrm{Drv}(\underline{\#\bot}, v)$.

   (a) Show that for any φ, if $\mathbb{N} \vDash \exists v\ \mathrm{Drv}(\underline{\#\varphi}, v)$, then $PA \vdash F$.
   (b) Show that $\mathbb{N} \vDash \mathrm{Con}(PA)$.
   (c) Give a consistent recursively enumerable theory T extending P_0, such that $T \vdash \mathrm{Con}(PA)$.

4. (4p) Show (in ZF) that AC is equivalent to the statement: For every set a whose elements are non-empty and pairwise disjoint, there is a set b such that for each $x \in a$, the intersection $b \cap x$ is a singleton.

5. (6p) The *busy beaver* function $\mathrm{BB} : \mathbb{N} \to \mathbb{N}$ is defined as follows: $\mathrm{BB}(n)$ is the largest number computable by some Turing machine with $\leq n$ bands and $\leq n$ states, on some input $\leq n$. This maximum is well-defined since there are only finitely many Turing machines and inputs within the given size bounds.

   (a) If $f : \mathbb{N} \to \mathbb{N}$ is a recursive function, show that the function $n \mapsto f(2n+1)+1$ is also recursive. You may assume the fact that the function $+ : \mathbb{N}^2 \to \mathbb{N}$ is recursive.

   (b) For T a Turing machine with a states and b bands ($b \geq 1$), and k any number, describe a Turing machine S with $a+k+1$ states and b band, such that running S from the initial configuration with all bands empty will have the same effect as running T on input k (i.e. S halts on blank input iff T halts on input k; and if they halt, they give the same output). *(Your description should be precise, but can be "procedural" rather than formal/symbolic — you don't need to formally write out transition functions etc.)*

   (c) Show that the busy beaver function is not recursive. *(Hint: use (a) and (b) to show that for any recursive $f : \mathbb{N} \to \mathbb{N}$, there is some j such that $BB(j) > f(j)$.)*

6. (6p) Let $\mathcal{L}_G$ be the language consisting of a single binary predicate symbol, $\sim$. An $\mathcal{L}_G$-structure (i.e. a set G together with a binary relation $\sim$) is called a *directed graph.*

   A *cycle of length* n in a directed graph $(G, \sim)$ (for $n \geq 1$) is a sequence $x_1, \ldots, x_n \in G$, such that $x_i \sim x_{i+1}$ for each $1 \leq i < n$, and $x_n \sim x_1$.

   (a) Show that the class of directed graphs containing no cycles is axiomatisable.

   (b) Show that the class of directed graphs containing at least one cycle is not axiomatisable.

7. (7p) A set x is *hereditarily countable* if its transitive closure $\mathrm{cl}(x)$ is countable. Write HC for the class of hereditarily countable sets.

   (a) Show that a set is hereditarily countable if and only if it is countable, and all its elements are hereditarily countable.

   (b) Show that $HC \subseteq V_{\omega_1}$. *(Hint: use the axiom of foundation.)*

   (c) Consider HC as a structure for the language of set theory (with the usual membership relation $\in$). Find an axiom of ZFC that does *not* hold in HC, and show it does not hold there. *(In fact, it can be shown that all except one of the axioms of ZFC hold in HC.)*