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Solutions to Written Exam
Logic II
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1. *Solution:*

(a) Informally the following function pd

$$\begin{aligned} \mathrm{pd}(0) &= 0 \\ \mathrm{pd}(x+1) &= x \end{aligned}$$

is defined by primitive recursion. It is immediate that $\mathrm{pd}(0) = 0$ and $\mathrm{pd}(x) = x - 1$ if $x > 0$.

(b) The definition of $\dot{-} : \mathbb{N}^2 \to \mathbb{N}$ is given by the following informal application of the primitive recursive scheme

$$\begin{aligned} x \dot{-} 0 &= x \\ x \dot{-} (y+1) &= \mathrm{pd}(x \dot{-} y) \end{aligned}$$

Thus $x \dot{-} y = \mathrm{pd}(\cdots \mathrm{pd}(x) \cdots) = \mathrm{pd}^y(x)$. If $x \geq y$, then $x = u + y$, for some u, so

$$x \dot{-} y = \mathrm{pd}^y(x) = \mathrm{pd}^y(u+y) = u = x - y,$$

and if $x \leq y$, then $y = u + x$, for some u, so

$$x \dot{-} y = \mathrm{pd}^y(x) = \mathrm{pd}^{u+x}(x) = \mathrm{pd}^u(0) = 0$$

- Remark. A completely formal definition of pd following Cori-Lascar (Ch 5.1) can be given as

$$\begin{aligned} \mathrm{pd}(0) &= \gamma_0() \\ \mathrm{pd}(x+1) &= P^2_1(x, \mathrm{pd}(x)) \end{aligned}$$

where $\gamma_0()$ is the 0-ary function with constant value 0. A formal definition of $\dot{-}$ is

$$\begin{aligned} x \dot{-} 0 &= P^1_1(x) \\ x \dot{-} (y+1) &= P^3_3(x, y, \mathrm{pd}(x \dot{-} y)) \end{aligned}$$

2. *Solution:*

(a) The set $E = \{(x, y) \in \mathbb{N}^2 : \phi^1_x = \phi^1_y\}$ is not recursive. Suppose it is recursive and that $\chi_E : \mathbb{N}^2 \to \mathbb{N}$ is its recursive characteristic function. Let e be an index so that ϕ^1_e is undefined for all inputs. Thus the set

$$\mathcal{C} = \{x \in \mathbb{N} : \phi^1_x = \phi^1_e\}$$

is recursive, since its characteristic function satisfies $\chi_{\mathcal{C}}(x) = \chi_E(x, e)$. Now we can see that $\mathcal{C}$ satisfies the conditions for Rice's Theorem. It is a set of indexes satisfying extensionality: if $x \in \mathcal{C}$, and $\phi^1_x = \phi^1_z$, then $z \in \mathcal{C}$. Moreover $\mathcal{C} \neq \emptyset$, since $e \in \mathcal{C}$. Let f be an index so that ϕ^1_f is e.g. the function that is constant 0. Thus $f \notin \mathcal{C}$, so $\mathcal{C} \neq \mathbb{N}$. Hence by Rice's theorem, $\mathcal{C}$ is not recursive. A contradiction. Hence E can not be recursive either.

(b) The set $F = \{x \in \mathbb{N} : W_x \text{ is finite}\}$ is not recursive. This follows again by an application of Rice's theorem, noting that F is a non-trivial index set. In fact, the indexes e and f from (a) above can be used to prove non-triviality: $e \in F$ and $f \in \mathbb{N} \setminus F$.

(c) The set $B = \{(x, y, n) \in \mathbb{N}^3$: Turing machine with index x stops on input y at the nth step $\}$ is recursive: Given (x, y, n) simulate the Turingmachine x with input y up to the nth step. This can be done using a primitive recursive function in x, y, n. Check whether the simulation has stopped before n step, then answer NO. If has stopped at step n, answer YES, else answer NO. (Cf. the ST function (Chapter 5.3.4, Cori-Lascar, vol. 2).)

3. *Solution:*

1. Cite for instance Definition 3.75 in Cori Lascar vol. 1

2. Consider the language $L_{\text{ord}} = \{\leq\}$. The same method as in Theorem 3.79 (Cori Lascar vol. 1) can be used.

3. Let PO be the theory $\{\forall x(x \leq x), \forall xy(x \leq y \wedge y \leq x \Rightarrow x = y), \forall xyz(x \leq y \wedge y \leq z \Rightarrow x \leq z)\}$. An L_{ord}-structure $\mathcal{A}$ is a partial order if and only if $\mathcal{A} \models \text{PO}$. Let F_n be the sentence that states that there are at least n different elements. Then $\text{PO} \cup \{F_n : n \geq\}$ axiomatizes infinite partial orders

4. *Solution:*

Let κ be a cardinal larger than $\max(\text{Card}(L), \text{Card}(\mathcal{M}))$. Apply the Upward Skolem-Löwenheim Theorem (8.15 in Cori-Lascar vol. 2) to obtain an L-structure $\mathcal{N}$ which is an elementary extension of $\mathcal{M}$ and whose cardinality is κ. Since $\kappa > \text{Card}(\mathcal{M})$, $\mathcal{N}$ cannot be isomorphic to $\mathcal{M}$. But since $\mathcal{M} \prec \mathcal{N}$, it holds that $\mathcal{M} \equiv \mathcal{N}$.

5. *Solution:*

Statements of the three equivalent forms of the axiom of choice can be found in Theorem 7.41 of Cori-Lascar vol. 2.

A suitable version for proving the order extension result is Zorn's Lemma. Solution: Let $P \subseteq S \times S$ a partial order on S. Form the set of all partial orders (p.o.) on S extending P:

$$R = \{Q \subseteq S \times S : \ Q \text{ p.o. on } S \text{ and } P \subseteq Q\}$$

This set is partially ordered by $\subseteq$. It is straightforward to verify that $(R, \subseteq)$ is an inductive set. By Zorn's Lemma $(R, \subseteq)$ has a maximal element L.

Next show that L is a total order on S. Suppose L is not a total order. Then there exists $x, y \in S$ such that $(x, y) \notin L$ and $(y, x) \notin L$. Next define

$$L^* = L \cup \{(u, v) \in S \times S : u \leq x, y \leq v\}.$$

Then reasoning by cases shows that L^* is a partial order. Clearly $L^* \supseteq L$ and $(x, y) \in L^*$. Hence L^* is properly including L contradictng that L was a maximal partial order in R. We conclude that L must be a total order.

6. *Solution:*

   1. $|S_1| = |\mathbb{R}^{\mathbb{Q}}| = |(2^{\mathbb{N}})^{\mathbb{Q}}| = |(2^{\mathbb{N}})^{\mathbb{N}}| = |2^{\mathbb{N}\times\mathbb{N}}| = |2^{\mathbb{N}}| = 2^{\aleph_0}$

   2. If $f, g : \mathbb{R} \to \mathbb{R}$ are two continuous functions such that $f_{|\mathbb{Q}} = g_{|\mathbb{Q}}$, then by continuity $f = g$. Thus $f \mapsto f_{|\mathbb{Q}}$ defines an injective function $S_2 \to S_1$. Hence $|S_2| \leq 2^{\aleph_0}$. Conversely, there is an injective map $\mathbb{R} \to S_2$, taking a real number to a constant map. Hence $2^{\aleph_0} = |\mathbb{R}| \leq |S_2|$. Thus $|S_2| = 2^{\aleph_0}$.

   3. $S_3 = \{f : \mathbb{N} \to \mathbb{N} \mid f \text{ is increasing}\}$. Clearly $S_3 \subseteq \mathbb{N}^{\mathbb{N}}$, so $|S_3| \leq {\aleph_0}^{\aleph_0} = 2^{\aleph_0}$. Conversely, if $h : \mathbb{N} \to \{0, 1\}$, then $f_h(n) = h(0) + \cdots + h(n)$ defines an increasing function $f_h \in S_3$. Now $h \mapsto f_h$ defines an injective function $\{0, 1\}^{\mathbb{N}} \to S_3$. Thus also $2^{\aleph_0} \leq |S_3|$. Hence $|S_3| = 2^{\aleph_0}$.

   4. By a non-decreasing function $f : \mathbb{N} \to \mathbb{N}$ is meant a function which has the property that $f(n) \geq f(n+1)$ for all n. Since values are taken in $\mathbb{N}$, which is well ordered, there must exist a k such that $f(k) = f(k+n)$ for all $n > 1$. Thus a function $f \in S_4$ is determined by a sequence $f(0), \ldots, f(k)$, where k is smallest such that $f(k) = f(k+n)$ for all $n > 1$. It follows that

   $$|S_4| = |\cup_{n\geq 1} \mathbb{N}^k| = |\mathbb{N}| = \aleph_0.$$

7. *Solution:*
   Gödel's First Incompleteness Theorem is stated in 6.30 of Cori-Lascar vol 2. We use it to prove that the theory

$$T = \mathrm{PA} \cup \{F_n : n > 2\}$$

is incomplete. We need only to note that

   (a) it includes the theory PA_0,

   (b) it is consistent, since $\mathbb{N} \models \mathrm{PA}$, and by Wiles' theorem $\mathbb{N} \models F_n$ for each $n > 2$

   (c) and T is recursive since PA is recursive and there is a (primitive) recursive function which can decide whether a formula G satisfies $\#F_n = \#G$ for some $n > 2$.

8. *Solution:*

   1. Let $\lambda(u)$ be the L_{ring}-formula

   $$\exists x\, \exists y\, \exists z\, \exists w\, (u = x^2 + y^2 + z^2 + w^2).$$

   By Lagrange's theorem, for all $u \in \mathbb{Z}$, $\mathcal{Z} \models \lambda(u)$ if, and only if, $u \geq 0$. So $\mathbb{N}$ is defined by $\phi(u)$ in $\mathcal{Z}$.

   2. We know by Gödel's theorem (6.28 Cori-Lascar vol 2) that

   $$\mathrm{Th}(\mathcal{N}) = \{\#F : \ \mathcal{Z} \models F \text{ and } F \text{ is a closed } L_{\mathrm{ring}}\text{-formula}\}$$

is not recursive, since it is consistent and contains PA_0 ($\mathcal{P}_0$).

With the help of (a) we can translate first order decision problems of $\mathcal{Z}$ to a decision problems of $\mathcal{N}$ by restricting quantifiers to $\mathbb{N}$.

Define for each L_{ring}-formula F, an L_{ring}-formula F^λ (F *relativized to* λ):

- $F^\lambda = F$ if F atomic
- $(F \wedge G)^\lambda = F^\lambda \wedge G^\lambda$
- $(F \vee G)^\lambda = F^\lambda \vee G^\lambda$
- $(F \Rightarrow G)^\lambda = F^\lambda \Rightarrow G^\lambda$
- $(\forall x\, F)^\lambda = \forall x(\lambda(x) \Rightarrow F^\lambda)$
- $(\exists x\, F)^\lambda = \exists x(\lambda(x) \wedge F^\lambda)$

One can show by induction on F that for all $a_1, \ldots, a_n \in \mathbb{N}$,

$$\mathcal{Z} \models F^\lambda[a_1, \ldots, a_n] \text{ iff } \mathcal{N} \models F[a_1, \ldots, a_n],$$

and in particular for closed F,

$$\mathcal{Z} \models F^\lambda \text{ iff } \mathcal{N} \models F.$$

It is relative straightforward to see that there is a (primitive) recursive function $f : \mathbb{N} \to \mathbb{N}$, acting on the Gödel coding so that $f(\#F) = \#(F^\lambda)$ for L_{ring}-formula F.

Thus $\#F \in \mathrm{Th}(\mathcal{N})$ iff $f(\#F) = \#(F^\lambda) \in \mathrm{Th}(\mathcal{Z})$. Therefore if $\mathrm{Th}(\mathcal{Z})$ is recursive, then so is $\mathrm{Th}(\mathcal{N})$. A contradiction. Hence $\mathrm{Th}(\mathcal{Z})$ is not recursive.
