STOCKHOLM UNIVERSITY
Department of Mathematics
Examiner: Gregory Arone

Examination for
MM3001: Matematiska metoder för ekonomer
15th March 2021

Time: 8:00-13:00
Instructions:
- During the exam you MAY NOT use textbooks, class notes, or any other supporting material.
- Use of calculators is permitted for performing calculations. The use of graphic or programmable features is NOT permitted.
- In all of your solutions, give explanations to clearly show your reasoning. Points may be deducted for unclear solutions even if the answer is correct.
- Use natural language when appropriate, not just mathematical symbols.
- Write clearly and legibly.
- Where applicable, indicate your final answer clearly by putting A BOX around it.
- The solutions should be uploaded onto the course's webpage no later than 13:30

Note: There are six problems, some with multiple parts. The problems are not ordered according to difficulty

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1. Let k be a fixed number. Consider the following system of linear equations, with unknowns x, y, z, and w.

$$\begin{aligned} 3x + y - 2z + w &= 5 \\ x - y - z + w &= 6 \\ 5x + 3y - 3z + kw &= 4 \end{aligned}$$

   (a) Use Gaussian elimination to find for which value of k the system of equations has at least one solution. (2p)

   Solution: Let us write the augmented matrix of coefficients, and apply Gaussian elimination

$$\begin{bmatrix} 3 & 1 & -2 & 1 & 5 \\ 1 & -1 & -1 & 1 & 6 \\ 5 & 3 & -3 & k & 4 \end{bmatrix}$$

R1 $\mapsto \frac{1}{3}$R1.

$$\begin{bmatrix} 1 & \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} & \frac{5}{3} \\ 1 & -1 & -1 & 1 & 6 \\ 5 & 3 & -3 & k & 4 \end{bmatrix}$$

R2 $\mapsto$R2$-$R1, R3 $\mapsto$R3 $-$ 5R1.

$$\begin{bmatrix} 1 & \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} & \frac{5}{3} \\ 0 & -\frac{4}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{13}{3} \\ 0 & \frac{4}{3} & \frac{1}{3} & k-\frac{5}{3} & -\frac{13}{3} \end{bmatrix}$$

R2 $\mapsto -\frac{3}{4}$R2

$$\begin{bmatrix} 1 & \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} & \frac{5}{3} \\ 0 & 1 & \frac{1}{4} & -\frac{1}{2} & -\frac{13}{4} \\ 0 & \frac{4}{3} & \frac{1}{3} & k-\frac{5}{3} & -\frac{13}{3} \end{bmatrix}$$

R1 $-\frac{1}{3}$R2, R3 $-\frac{4}{3}$R2

$$\begin{bmatrix} 1 & 0 & \frac{-3}{4} & \frac{1}{2} & \frac{11}{4} \\ 0 & 1 & \frac{1}{4} & -\frac{1}{2} & -\frac{13}{4} \\ 0 & 0 & 0 & k-1 & 0 \end{bmatrix}$$

At this point we can conclude that the system $\boxed{\text{has a solution for every } k}$. However, the number of degrees of freedom depends on whether $k \neq 1$ or $k = 1$. If $k \neq 1$, then we can divide R3 by $k-1$, clean up the rest of the fourth column, and the matrix has the following final form

$$\begin{bmatrix} 1 & 0 & \frac{-3}{4} & 0 & \frac{11}{4} \\ 0 & 1 & \frac{1}{4} & 0 & -\frac{13}{4} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and the general solution has one degree of freedom. On the other hand, if $k = 1$ then the augmented coefficients matrix has the following form

$$\begin{bmatrix} 1 & 0 & \frac{-3}{4} & \frac{1}{2} & \frac{11}{4} \\ 0 & 1 & \frac{1}{4} & -\frac{1}{2} & -\frac{13}{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the general solution has two degrees of freedom.

(b) For the value of k that you found in part (a), describe the general solution. Your answer should express x and y in terms of z and w. (2p)

Solution: If $k \neq 1$ then the general solution has the following form

$$\begin{aligned} x &= \tfrac{11}{4} + \tfrac{3}{4}z \\ y &= -\tfrac{13}{4} - \tfrac{1}{4}z \\ w &= 0 \end{aligned} .$$

If $k = 1$ then the general solution has the following form.

$$\begin{aligned} x &= \tfrac{11}{4} + \tfrac{3}{4}z - \tfrac{1}{2}w \\ y &= -\tfrac{13}{4} - \tfrac{1}{4}z + \tfrac{1}{2}w \end{aligned} .$$

(c) Find the solution with $z = -1$, $w = 2$. (1p)

Solution: $\boxed{\text{If } k \neq 1 \text{ there is no such solution. If } k = 1 \text{ then } (x, y, z, w) = (1, -2, -1, 2)}$

2. Consider the equation

$$y^2x^2 + \frac{x}{\sqrt{y}} = 6.$$

This equation defines a curve in the plane. Notice that $(2, 1)$ is a solution

(a) Use implicit differentiation to find the slope of the tangent line to this curve at the point $(2, 1)$. (3p)

Solution: Differentiating the equation, while viewing y as a function of x gives us the following equation:

$$2yy'x^2 + 2y^2x + \frac{1}{\sqrt{y}} - \frac{xy'}{2\sqrt[3]{y^2}} = 0.$$

Which can be rewritten as

$$y'(2yx^2 - \frac{x}{2\sqrt[3]{y^2}}) + (2y^2x + \frac{1}{\sqrt{y}}) = 0.$$

It follows that

$$y' = \frac{2y^2x + \frac{1}{\sqrt{y}}}{\frac{x}{2\sqrt[3]{y^2}} - 2yx^2}.$$

Substituting $x = 2, y = 1$ we obtain the answer $\boxed{y' = -\frac{5}{7}}$.

(b) Find the equation of the tangent line at the point $(2, 1)$. (2p)

Solution: This is a line of slop $-\frac{5}{7}$ passing through the point $(2, 1)$. Therefore its equation is

$$\frac{y-1}{x-2} = -\frac{5}{7}.$$

Simplifying, we get the equation $\boxed{5x + 7y = 17}$.

3. (a) Compute the integral $\int (t^2+1)e^{t^3+3t}\, dt$ (as a function of t). (2p)

Solution: Use change of variables $u = t^3 + 3t$. We get $du = (3t^2 + 3)dt$, or $(t^2 + 1)dt = \frac{du}{3}$. Therefore

$$\int (t^2+1)e^{t^3+3t}\, dt = \int \frac{e^u}{3}\, du = \frac{e^u}{3} + C = \boxed{\frac{e^{t^3+3t}}{3} + C}.$$

(b) Find a number a for which $\int_a^0 \sqrt{1-x}\, dx = \frac{14}{3}$ (3p).

Solution: Use the substitution $u = 1 - x$, $du = -dx$

$$\int_a^0 \sqrt{1-x}\, dx = \int_{1-a}^1 \sqrt{u} \cdot (-du) = \int_1^{1-a} \sqrt{u}\, du = \left.\frac{2u^{\frac{3}{2}}}{3}\right]_1^{1-a} = \frac{2}{3}\left((1-a)^{\frac{3}{2}} - 1\right).$$

We obtain the equation

$$\frac{2}{3}\left((1-a)^{\frac{3}{2}} - 1\right) = \frac{14}{3}.$$

It simplifies to

$$(1-a)^{\frac{3}{2}} = 8.$$

So $1-a = 8^{\frac{2}{3}} = 4$ and $\boxed{a=-3}$.

4. Let a be some fixed number. Consider the function $f(x,y) = x^2 + axy + y^2 - 4x - ax - 2y - 2ay$.
   (a) Prove that $(2,1)$ is a critical point of f, for every a. (2p)

   Solution: we calculate the partial derivatives of f

   $$f'_x(x,y) = 2x + ay - 4 - a, \quad f'_y(x,y) = ax + 2y - 2 - 2a.$$

   Substituting $x=2, y=1$ we obtain

   $$f'_x(2,1) = 4 + a - 4 - a = 0, \quad f'_y(2,1) = 2a + 2 - 2 - 2a = 0.$$

   So $(2,1)$ is a critical point, regardless of a.

   (b) Find the second derivatives f''_{xx}, f''_{xy} and f''_{yy}. You answer may depend on a. (1p)

   Solution:

   $$f''_{xx}(x,y) = 2, \quad f''_{xy}(x,y) = a, \quad f''_{xy}(x,y) = 2.$$

   (c) Find for which a (if any) the point $(2,1)$ is a local maximum, for which a it is a local minimum, and for which it is neither. [The formula at the end of the test may help.] (2p)

   Solution: We have the expression $f''_{xx}f''_{yy} - (f''_{xy})^2 = 4 - a^2$. By the second derivative criterion, and the convexity criterion we obtain that

   - $\boxed{\text{if } a > 2 \text{ or } a < -2 \text{ then}}$ $f''_{xx}f''_{yy} - (f''_{xy})^2 < 0$, and the critical point is **neither a maximum nor minimum**.
   - $\boxed{\text{If } -2 \le a \le 2 \text{ then}}$ $f''_{xx}f''_{yy} - (f''_{xy})^2 \ge 0$ for all x, y. Since $f''_{xx} = 2 > 0$, f is convex on the entire plane, and **the critical point is a local (and in fact global) minimum** .

5. Consider the function

   $$f(x,y) = 3x^2 - 12x + 3y^2 - 4y.$$

   Let D be the domain defined by the inequalities $0 \le y$ and $x^2 + y^2 \le 10$.

   Find the global maximum and the global minimum of $f(x,y)$ on D. Remember to show clearly all the necessary steps. (5p)

   Solution: We have to prepare our list of suspects. We proceed to find

   Step one: interior points. We look for the critical points

$$f'_x(x,y) = 6x - 12 = 0, \quad f'_y(x,y) = 6y - 4 = 0.$$

We obtain a single critical point $(2, \frac{2}{3})$. It is easily checked that it is inside D, so it goes onto the list of suspects.

Step two: points satsifying $y = \sqrt{10 - x^2}$. We substitute this expression into the function to obtain

$$f(x, \sqrt{10 - x^2}) = 3x^2 - 12x + 3(10 - x^2) - 4\sqrt{10 - x^2} = 30 - 12x - 4\sqrt{10 - x^2}.$$

to find the critical points of this function we differentiate it and look where the derivative is zero

$$-12 - 4\frac{-2x}{2\sqrt{10 - x^2}} = -12 + 4\frac{x}{\sqrt{10 - x^2}} = 0.$$

This simplifes to

$$\frac{x}{\sqrt{10 - x^2}} = 3, \quad x^2 = 90 - 9x^2, \quad x^2 = 9, \quad x = \pm 3.$$

But the derivative is zero only for $x = 3$. Since $y = \sqrt{10 - x^2}$, we obtain a critical point $(3, 1)$.

Step three: points satsifying $y = 0$, and therefore $-\sqrt{10} \leq x \leq \sqrt{10}$. We substitute $y = 0$ into the function to obtain

$$f(x, 0) = 3x^2 - 12x.$$

The derivative of this function is $6x - 12$. Setting it equal zero, we obtain the critical point $x = 2$, so $(2, 0)$ goes onto the list of suspects.

Step four: points where the boundary is not smooth: $(-\sqrt{10}, 0)$ and $(\sqrt{10}, 0)$.

Summarizing, we obtain the following table of suspect points and values of f:

(x,y)	$f(x,y)$
$(2, \frac{2}{3})$	$-13\frac{1}{3}$
$(3, 1)$	-10
$(2, 0)$	-12
$(-\sqrt{10}, 0)$	$30 + 12\sqrt{10} \approx 68$
$(\sqrt{10}, 0)$	$30 - 12\sqrt{10} \approx -8$

From this table we read that

f has a maximum value $30 + 12\sqrt{10} \approx 68$ at $(-\sqrt{10}, 0)$ and a minimal value $-13\frac{1}{3}$ at $(2, \frac{2}{3})$.

6. Consider the function $f(x) = \sqrt{\ln(x^2 - x - 2)}$.

   (a) Determine the domain of definition of f. (2p)

   Solution: The function is well-defined when $\ln(x^2 - x - 2) \geq 0$. This is equivalent to saying that $x^2 - x - 2 \geq 1$, or $x^2 - x - 3 \geq 0$. We solve the associated quadratic equation

$$x_{1,2} = \frac{1 \pm \sqrt{13}}{2}.$$

It follows that the domain of definition is $\boxed{(-\infty, \frac{1-\sqrt{13}}{2}] \cup [\frac{1+\sqrt{13}}{2}, \infty)}$.

(b) Determine the local extreme points of f (if any). (1p)

Solution: We differentiate, and look for critical points. So we obtain the equation

$$f'(x) = \frac{2x-1}{2(x^2-x-2)\sqrt{\ln(x^2-x-2)}} = 0.$$

We saw that for all x in the domain of the function, $x^2 - x - 2 \geq 1$, and in particular $x^2 - x - 2 > 0$. It follows that the denominator is positive for all x in the domain of f. There is a potential critical point at $x = \frac{1}{2}$, but it is not in the domain of definition. So f is a differentiable function with no critical points, and therefore it has $\boxed{\text{no local extreme points}}$.

Remark: In fact, the endpoints of the intervals $\frac{1-\sqrt{13}}{2}$ and $\frac{1+\sqrt{13}}{2}$ are local minima, but in this class we identify "local extereme points" with critical points, and I don't expect the students to see this.

(c) Determine where f is increasing and where f is decreasing. (2p)

Solution: Consider again the expression for $f'(x)$. We say that the denominator is always positive. It follows that f is increasing whenever the numerator $2x - 1$ is positive and is decreasing whenever it is negative. It follows that f is

$\boxed{\text{decreasing on } (-\infty, \frac{1-\sqrt{13}}{2}) \text{ and increasing on } (\frac{1+\sqrt{13}}{2}, \infty)}$.

Formulas

The second derivative criterion for a function of two variables $f(x, y)$ depends on the determinant $\det \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{bmatrix}$. It says the following: If, at a critical point

- $\det \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{bmatrix} > 0$ and $f''_{xx} > 0$ then f has a local minimum at this critical point.
- $\det \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{bmatrix} > 0$ and $f''_{xx} < 0$ then f has a local maximum at this critical point.
- $\det \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{bmatrix} < 0$ then f has neither a local maximum nor a local minimum at this critical point.

GOOD LUCK!