STOCKHOLM UNIVERSITY
Department of Mathematics
Examiner: Gregory Arone

Examination for
MM3001: Matematiska metoder för ekonomer
14th April 2021

Time: 8:00-13:00
Instructions:
- During the exam you MAY NOT use textbooks, class notes, or any other supporting material.
- Use of calculators is permitted for performing calculations. The use of graphic or programmable features is NOT permitted.
- Start every problem on a new page, and write at the top of the page which problem it belongs to. (But in multiple part problems it is not necessary to start every part on a new page)
- In all of your solutions, give explanations to clearly show your reasoning. Points may be deducted for unclear solutions even if the answer is correct.
- Use natural language when appropriate, not just mathematical symbols.
- Write clearly and legibly.
- Where applicable, indicate your final answer clearly by putting [A BOX] around it.
- The solutions should be uploaded onto the course's webpage no later than 13:30

Note: There are six problems, some with multiple parts. The problems are not ordered according to difficulty

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1. (a) Suppose f is the following function

$$f(x) = \int_0^x e^{t^2}\, dt. \qquad \text{(2p)}$$

What is $f'(x)$? Hint: do not try to evaluate the integral.

Answer: By the fundamental theorem of calculus,

$$f'(x) = e^{x^2}.$$

(b) Now suppose g is the following function (defined for $x > 0$)

$$g(x) = \int_0^{\sqrt{x}} e^{t^2}\, dt \qquad \text{(3p)}$$

What is $g'(x)$?

Answer: $g(x) = f(\sqrt{x})$ and therefore by the chain rule

$$g'(x) = f'(\sqrt{x}) \cdot (\sqrt{x})' = \frac{e^{\sqrt{x}^2}}{2\sqrt{x}} = \frac{e^x}{2\sqrt{x}}.$$

2. Suppose that $f(x), g(x)$ are two functions, and $h(x) = f(g(x))$. Assume furthermore that

$$f(2) = 4, \quad f'(2) = -2, \quad g(3) = 2, \quad g'(3) = -1.$$

(a) What is the equation of the tangent line to the graph of f at the point $(2, f(2))$? (2p)

Answer: It is a line passing through $(2, 4)$ of slope -2, so its equation is

$$\frac{y-4}{x-2} = -2.$$

Or in more standard form

$$y = -2x + 8.$$

(b) What is $h(3)$? (1p)
Answer: $h(3) = f(g(3)) = f(2) = 4$.

(c) What is $h'(3)$? (2p)
Answer: By the chain rule, since $h(x) = f(g(x))$, $h'(x) = f'(g(x))g'(x)$ so

$$h'(3) = f'(g(3)) \cdot g'(3) = f'(2) \cdot g'(3) = (-2) \cdot (-1) = 2.$$

3. (a) Compute the improper integral $\displaystyle\int_{e^2}^{\infty} \frac{1}{x(\ln x)^2}\, dx$. (3p)
Answer: We use the substitution $u = \ln x$, $du = \frac{dx}{x}$. Notice what as x goes from e^2 to infinity, u goes from 2 to infinity.

$$\int_{e^2}^{\infty} \frac{1}{x(\ln x)^2}\, dx = \int_2^{\infty} \frac{1}{u^2}\, du = \lim_{c\to\infty} \int_2^c \frac{1}{u^2}\, du = \lim_{c\to\infty} [-\frac{1}{u}]_2^{\infty} = \lim_{c\to\infty} \frac{1}{2} - \frac{1}{c} = \frac{1}{2}.$$

(b) Let a be a fixed number, and let $f(x)$ be the following function, depending on a

$$f(x) = \begin{cases} \sqrt{ax} & x > 2 \\ a - x & x \leq 2 \end{cases}$$

For which value(s) of a is this function continuous? (2p).
Answer: The function is continuous when both formulas agree at 2. This means that a has to satisfy the following equation

$$\sqrt{2a} = a - 2$$

This equation implies that

$$2a = (a-2)^2 = a^2 - 4x + 4$$

or equivalently

$$a^2 - 6a + 4 = 0$$

Solving the quadratic equation we find that

$$a_{1,2} = 3 \pm \sqrt{5}$$

Notice that $3 - \sqrt{5} < 2$, therefore this number does not satisfy the equation $\sqrt{2a} = a - 2$. So the only solution is $\boxed{a = 3 + \sqrt{5}}$.

4. Consider the function of two variables: $f(x, y) = x^2y - xy$

   (a) Find the critical points of this function. (3p)
   Answer: The critical points are points (x, y) that satsify *both* of the following equations.

$$\begin{aligned} f'_x(x,y) = 2xy - y &= 0 \\ f'_y(x,y) = x^2 - x &= 0 \end{aligned}$$

   The first equation says that either $x = \frac{1}{2}$ or $y = 0$. The second equation says either $x = 0$ or $x = 1$. It follows that there is no critical point with $x = \frac{1}{2}$. The only critical points are $(0, 0)$ and $(1, 0)$.

   (b) For each one of the critical points, determine if it is a local maximum, a local minimum, or neither. (2p)
   Answer: First, let us calculate the second derivatives

$$f''_{xx} = 2y, \quad f''_{xy} = 2x - 1, \quad f''_{yy} = 0.$$

   Next, we need to calculate the determinant of the matrix of second derivatives.

$$\det \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{bmatrix} = \det \begin{bmatrix} 2y & 2x-1 \\ 2x-1 & 0 \end{bmatrix} = -(2x-1)^2$$

   At both critical points, the determinant is -1, which is a negative number. It follows that each critical point is a saddle point, i.e., neither a local minumum nor a local maximum.

5. Consider the function

$$f(x) = \sqrt{x^2 + x + 2}.$$

   (a) Find the domain of definition of f. (1p)
   Answer: The function is defined whenever $x^2 + x + 2 \geq 0$. It is easy to check that this holds for all x, so the domain is all of $\mathbb{R}$.

   (b) Find the intervals where f is increasing and where f is decreasing. (2p)
   Answer:

$$f'(x) = \frac{2x + 1}{2\sqrt{x^2 + x + 2}}.$$

   The denominator is always positive, so the sign of $f'(x)$ is the same as the sign of $2x+1$. It follows that the function is increasing on the interval $[-\frac{1}{2}, \infty)$ and decreasing on $(-\infty, -\frac{1}{2}]$.

   (c) Find the minimum and the maximum of f on the interval $[-5, 5]$. (2p)
   Answer: by the calculation of $f'(x)$, the only critical point of f is $x = -\frac{1}{2}$, which is inside the interval. So our list of suspects consists of $-5, -\frac{1}{2}$, and 5. Evaluating f at these points we find that

$$\begin{aligned} f(-5) &= \sqrt{22} \\ f\left(-\frac{1}{2}\right) &= \frac{\sqrt{7}}{2} \\ f(5) &= \sqrt{32} \end{aligned}$$

It follows that f attains a minimum at $x = -\frac{1}{2}$ and a maximum at $x = 5$.

6. Let a be a fixed number. Consider the following system of equations

$$\begin{aligned} x + 2y + 3z &= 2 \\ x - y + 2z &= 1 \\ x + (10 - a)y + 5z &= 3 \end{aligned}$$

Use Gaussian elimination to:

(a) Find for which values of a (if any) the system has a unique solution, for which it has no solutions and for which it has infinitely many solutions. (2p)

Answer: Let us write the extended matrix of coefficients

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & -1 & 2 & 1 \\ 1 & 10-a & 5 & 3 \end{bmatrix}$$

Now we perform the Gauss elimination, indicating at each step the row operations. We begin with the operations R2 →R2−R1 and R3 →R3−R1

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -3 & -1 & -1 \\ 0 & 8-a & 2 & 1 \end{bmatrix}$$

R3 $\rightarrow -\frac{1}{3}$R3.

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 8-a & 2 & 1 \end{bmatrix}$$

R1 →R1 − 2R2, $R3$ →R3 − $(8 - a)$R2

$$\begin{bmatrix} 1 & 0 & \frac{7}{3} & \frac{4}{3} \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 2 - \frac{8-a}{3} & 1 - \frac{8-a}{3} \end{bmatrix}$$

Simplifying, we obtain the following matrix

$$\begin{bmatrix} 1 & 0 & \frac{7}{3} & \frac{4}{3} \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{a-2}{3} & \frac{a-5}{3} \end{bmatrix}$$

Notice that if $a = 2$ we obtain in the last line the equation $0 = -1$, which is impossible. We conclude that if $a = 2$ then the system has no solutions. Assuming that $a \neq 2$ we divide the third row by $\frac{a-2}{3}$ to obtain the following matrix

$$\begin{bmatrix} 1 & 0 & \frac{7}{3} & \frac{4}{3} \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{a-5}{a-2} \end{bmatrix}$$

Finally, we perform the operations R1 →R1 $- \frac{7}{3}$R3 and R2 $- \frac{1}{3}$R3

$$\begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} - \frac{7}{3}\frac{a-5}{a-2} \\ 0 & 1 & 0 & \frac{1}{3} - \frac{a-5}{3(a-2)} \\ 0 & 0 & 1 & \frac{a-5}{a-2} \end{bmatrix}$$

Simplifying, we obtain the following matrix

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{a-9}{a-2} \\ 0 & 1 & 0 & \frac{1}{a-2} \\ 0 & 0 & 1 & \frac{a-5}{a-2} \end{bmatrix}$$

We conclude that when $a \neq 2$, the system has a unique solution.

(b) In cases when there is a unique solution, express the solution in terms of a. (3p)

Answer: From the matrix at the end of part (a) we conclude that when $a \neq 2$ the unique solution is given by

$$x = \frac{9-a}{a-2}, \quad y = \frac{1}{a-2}, \quad z = \frac{a-5}{a-2}.$$

Formulas

The second derivative criterion for a function of two variables $f(x, y)$ depends on the determinant $\det \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{bmatrix}$. It says the following: If, at a critical point

- $\det \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{bmatrix} > 0$ and $f''_{xx} > 0$ then f has a local minimum at this critical point.
- $\det \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{bmatrix} > 0$ and $f''_{xx} < 0$ then f has a local maximum at this critical point.
- $\det \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{bmatrix} < 0$ then f has neither a local maximum nor a local minimum at this critical point.

GOOD LUCK!