Department of mathematics,
Stockholm University
Examinator: Gregory Arone

MM7041–Topology
Final Exam

Jan 7, 2021
13:00-18:00
At home exam

Instructions: Work alone. You are allowed to use the textbook and the class notes. You can quote results from the textbook and from the class, but state clearly which result you are using. You are **not allowed** to search the internet for solutions or hints.

Justify all your answers with a proof or a counterexample. A simple Yes or No answer, even if correct, may get partial or no credit.

Problems have multiple parts. In some cases, later parts depend on earlier ones. Even if you could not do the earlier parts, you **are allowed** to use the results of the earlier parts in the later parts.

On the first page, please have the following:

- Name
- Social security number
- Write out and sign the following pledge: *On my honor as a student I have not received help or used inappropriate resources on this exam.*
- List the problems that you have attempted.

Start each problem on a new page (but it is not necessary to start each part of a problem on a new page). Write at the top of each page which problem it belongs to.

1. Let $\mathbb{N}$ be the set of positive integers. Let $\mathcal{T}$ be the collection of subsets $U \subset \mathbb{N}$ satisfying the following: if $x \in U$ and x is odd, then also $x+1 \in U$.
   (a) [2 pts] Prove that $\mathcal{T}$ is a topology on $\mathbb{N}$.
   For the rest of this problem, $\mathbb{N}$ denotes the set of positive integers with the topology $\mathcal{T}$.
   (b) [1 pt] Describe the closed subsets of $\mathbb{N}$.
   (c) [2 pts] Describe the connected components of $\mathbb{N}$.
   (d) [1 pt] Describe the path components of $\mathbb{N}$.
   (e) [1 pt] Is $\mathbb{N}$ locally connected?

2. Define an equivalence relation on $\mathbb{R}^n \times \{1, 2\}$ by declaring that $(\bar{x}, 1) \sim (\bar{x}, 2)$ for all **nonzero** vectors $\bar{x} \in \mathbb{R}^n \setminus \{\bar{0}\}$. Let X be the quotient of $\mathbb{R}^n \times \{1, 2\}$ by this relation, and let $q \colon \mathbb{R}^n \times \{1, 2\} \to X$ be the quotient map.
   (a) [2 pts] Is q an open map?
   (b) [1 pt] Is q a closed map?
   (c) [2 pts] Is X locally Euclidean?
   (d) [1 pt] Is X Hausdorff?

3. Let $f \colon X \to Y$ be a continuous map. Recall that a retraction of f is a continuous map $r \colon Y \to X$ such that $r \circ f$ is the identity on X. Similarly, a section of f is a continuous map $s \colon Y \to X$ such $f \circ s$ is the identity on Y.

   Determine whether

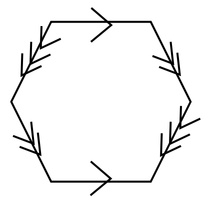
(a) [1 pt] The inclusion $(1,2) \hookrightarrow [0,3]$ has a retraction.

(b) [1 pt] The inclusion $[1,2] \to (0,3)$ has a retraction.

(c) [1 pt] The equatorial inclusion $S^1 \hookrightarrow S^2$ has a retraction.

(d) [2 pts] The squaring map $f(z) = z^2$ considered as a map from $\mathbb{C}$ to itself has a section.

4. Let X be the quotient space of the hexagon obtained by identifying opposite pairs of edges, as indicated in the following picture



(a) [1 pt] The standard CW structure on the hexagon, with six 0-cells, six 1-cells and a single 2-cell induces a CW-structure on X. How many cells in each dimension does X have with this structure? Draw a picture of the 1-skeleton of X.

(b) [1 pt] What is the Euler characteristic of X?

(c) [1 pt] Is X a surface? If yes, identify it with a familiar surface, if not, explain why it is not.

(d) [1 pt] Describe the fundamental group of X.

5. (a) [2 pts] Prove that the space $\mathbb{R}^4 \setminus \{\bar{0}\}$ is simply-connected.

Now consider the homeomorphism $f\colon \mathbb{R}^4 \setminus \{\bar{0}\}) \to \mathbb{R}^4 \setminus \{\bar{0}\}$ defined by the formula $f(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$. Notice that $f \circ f \circ f \circ f$ is the identity map.

(b) [2 pts] Prove that f induces a covering space action of $\mathbb{Z}/4$ on $R^4 \setminus \{\bar{0}\}$. In other words, prove that there is a covering space action of $\mathbb{Z}/4$ on $R^4 \setminus \{\bar{0}\}$, where the generator of $\mathbb{Z}/4$ acts by the map f.

Let $X = (R^4 \setminus \{\bar{0}\})/_{\mathbb{Z}/4}$ be the quotient space of this action. Remember that even if you did not do parts (a) and (b) of the problem, you still can assume them in the remaining parts.

(c) [1 pt] Find the fundamental group of X.

(d) [1 pt] How many pairwise non-isomorphic covering spaces does X have?

(e) [2 pts] Describe the automorphism group of each of the covering spaces of X that you found in the previous part.