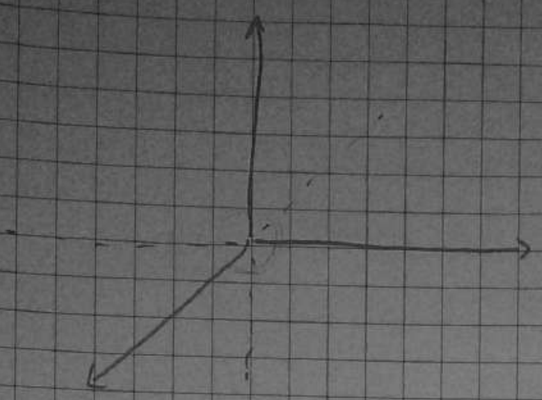
Solution for the exam May 2019

Exercise 1



(a) It is not open. ~~The sequence~~ ~~Let~~ $(0,0,0) \in X$

let ~~B~~ $Q_{\varepsilon} = (-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)$ with $\varepsilon \in \mathbb{R}^+$

~~As $\varepsilon \to 0$ then~~ The ~~f~~ collection $\{Q_{\varepsilon}\}_{\varepsilon \in \mathbb{R}^+}$ is a ~~b~~neighbouring basis of $(0,0,0)$

the point $\left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right) \in Q_{\varepsilon} \cap X^c \neq \emptyset$.

(b) $X^c = (-\infty,0)\times(0,+\infty)\times(0,+\infty) \cup$
$(0,\infty)\times(-\infty,0)\times(0,+\infty) \cup$
$(0,+\infty)\times(0,+\infty)\times(-\infty,0) \cup$
$(-\infty,0)\times(-\infty,0)\times(-\infty,0)$

these are all open in the Euclidean topology

So X^c is open and X is closed

(c) The boundary is $\partial X = \{(x,y,z) \mid xyz = 0\}$

~~Let $(x,y,z) \in \mathbb{R}^3$ if it is in~~

Note that the ~~product~~ sign of xyz is constant in the interior of a quadrant. If (x,y,z) is in ~~the~~ interior of a quadrant $P=(x,y,z) \in X$, then there is a neighbourhood of P such that is all contained in the quadrant.

then this neighbourhood is all contained in X and $P \notin \partial X$

(Similarly we can exclude that $P \notin X$ in the interior of ~~the~~ quadrant is in ∂X

We deduce that $\partial X \subseteq \{xyz = 0\}$

Now ~~suppose~~ take

$P = (x, y, 0) \in \{xyz = 0\}$ with $x, y \neq 0$

~~Con~~ The sets ~~[illegible]~~

$V_\varepsilon = (x-\varepsilon, x+\varepsilon) \times (y-\varepsilon, y+\varepsilon) \times (-\varepsilon, \varepsilon)$

yield a neighbouring basis for P

~~[illegible]~~ $(x, y, \frac{\varepsilon}{2})$ and $(x, y, -\frac{\varepsilon}{2})$

belong to V_ε and ~~[illegible]~~ yields products with different signs.

So ~~this~~ $P \in \partial X$.

~~[illegible]~~ the case in which $x = 0$ or $y = 0$ is dealed similarly.

If more than a coordinate is 0 (for example $y = 0$ $z = 0$)

then we can take the points

$$(x \tfrac{\varepsilon}{2} \tfrac{\varepsilon}{2}) \quad (x, \tfrac{\varepsilon}{2}, -\tfrac{\varepsilon}{2})$$

(d) ~~X~~ is starred with center $(0\,0\,0)$:

Given $(x, y\, z) \in X$ ~~[illegible]~~$(tx\ ty\ tz) \in X$ $\forall t \in [0, 1]$

In particularly it is path connected and so connected

(e) $\mathrm{Int}(X) = \{(x, y, z) \mid x, y, z > 0\} \cup$

$\{(x, y, z) / x > 0\ y, z < 0\} \cup$

$\{(~~x~~ y\ x, y < 0, z > 0\} \cup \{x, z < 0, y > 0\}$

These are open and disjoint in the Euclidean topology so it is not connected.

(f) As we observed that it is starred wr a point we deduce that X is contractible.

$$i: \{(0,0,0)\} \longrightarrow X$$
$$(0,0,0) \longmapsto (0,0,0)$$
$$r: X \longrightarrow \{(0,0,0)\}$$
$$(x,y,z) \longmapsto (0,0,0)$$

yields homotopy equivalences.

$H(x,y,z,t) = (tx\ ty\ tz)$ is the required homotopy.

Exercise 2

(a) & (b) The properties of being connected and compact are preserved by continuous map. As the quotient map is surjective we have that Y is both connected & compact.

(c) Y is not T_2. Let $\star = \pi\left(\left(\frac{1}{3}, \frac{2}{3}\right)\right)$ with $\pi: I \to Y$ the quotient map

Let $V \subseteq Y$ a neighbourhood of $\pi\left(\frac{1}{3}\right)$ then $\pi^{-1}(V)$ is a neighbourhood of $\frac{1}{3}$ and has such

$$\pi^{-1}(V) \cap \left(\tfrac{1}{3}, \tfrac{2}{3}\right) \neq \emptyset$$

$$\Rightarrow V = \pi(\pi^{-1}(V)) \ni \star$$

Similarly any neighbourhood of $\pi\left(\frac{2}{3}\right)$ contains $\star$

So $\pi\left(\frac{1}{3}\right)$ and $\pi\left(\frac{2}{3}\right)$ do not admit disjoint neighbourhoods

d) Consider $* = p([\frac{1}{3}, \frac{2}{3}])$ with $p\ I \to Z$ the quotient map. Consider

$$f : Z \longrightarrow I$$

$$z \longmapsto \begin{cases} 3/2\, t & \text{if } z = p(t) \quad t \in [0, \frac{1}{3}] \\ \frac{1}{2} + \frac{3}{2}\left(t - \frac{2}{3}\right) & z = p(t) \quad t \in [\frac{2}{3}, 1] \end{cases}$$

the two definition domains intersect on $*$ which is closed $\Rightarrow$ the map is continuous by the gluing lemma. It is clearly bijective. Plus

$$f((a,b)) = \begin{cases} \left(\frac{3}{2}a, \frac{3}{2}b\right) & \text{if } b < \frac{1}{3} \\ \left(\frac{1}{2} + \frac{3}{2}\left(a - \frac{2}{3}\right), \frac{1}{2} + \frac{3}{2}\left(b - 2/3\right)\right) & \text{if } a \geq \frac{2}{3} \\ \left(\frac{3}{2}a, \frac{1}{2} + \frac{3}{2}\left(b - \frac{2}{3}\right)\right) & \text{otherwise} \end{cases}$$

In particular it is open. So f is an homeom.

e) As $Z \simeq I$ which is T_2

Z cannot be homeomorphic to Y

Exercise 3

$$\begin{bmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+x & b+cx+y \\ & 1 & c+z \\ & & 1 \end{bmatrix}$$

in particular we have that if we identify G with $\mathbb{R}^3$ by the homeomorphism

$$\varphi \begin{bmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{bmatrix} = (x, y, z)$$ we have that

the multiplication can be expressed with polynomial equation. Therefore it is continuos and G is a topological group.

The multiplication is not commutative

$$\begin{bmatrix} 1 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}$$

$$\neq$$

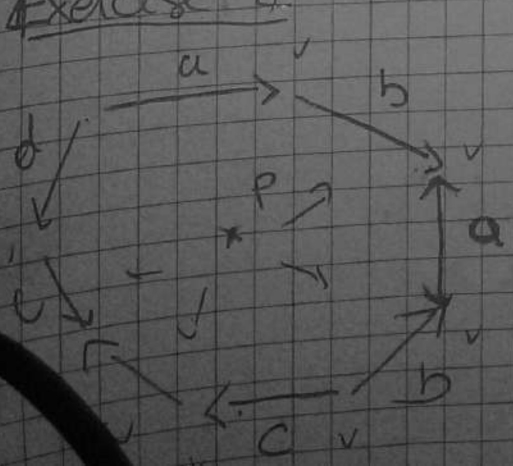
$$\begin{bmatrix} 1 & 0 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix}$$

So G is not isomorphic to $(\mathbb{R}^3, +)$

$$\begin{bmatrix} 1 & n & m \\ & 1 & k \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{bmatrix} = \left(m+x,\ m+\cancel{kz}+y,\ k+z\right)$$

if $V = I_x \times I_y \times I_z$ are interval neighbourhood of x, y, z respectively with I_x with diameter less than $\frac{1}{3}$ then $gV \cap V = \emptyset$ for every $g \in \Gamma$

Exercise 4



we have seen that this is homotopically equivalent to the boundary of the simplicial complex which is the ~~sa~~ wedge of 4 circles

SVK theorem yields

$$\pi_1(T\#T \setminus \{p\}) \simeq \mathbb{Z}^{*4}$$

Exercise 5

This is a theoretical question. We refer to the textbook.