MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Tentamensskrivning i
Matematik III Komplex Analys
7.5 hp
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No calculators, books, or other resources allowed. Max score on each problem is 5p; grade of E guaranteed at 15p. Appropriate amount of details required for full marks.

1. Determine the value of the integral

$$\int_0^\infty \frac{1}{(1+x^2)^3}\,\mathrm{d}x.$$

Solution: First note that the integrand is an even function. Hence,

$$\int_0^\infty \frac{1}{(1+x^2)^3}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{(1+x^2)^3}.$$

We rewrite the latter integral as the contour integral of the function $f(z) = \frac{1}{(1+z^2)^3}$ along a counter-clockwise parametrization of the contour $C_\rho = [-\rho,\rho] \cup C_\rho^+$, where C_ρ^+ denotes the upper half of the circle of radius ρ centered at zero. Thus

$$\int_{-\infty}^\infty \frac{1}{(1+z^2)^3}\mathrm{d}x = \lim_{\rho\to\infty}\int_{C_\rho} f(z)\mathrm{d}z - \lim_{\rho\to\infty}\int_{C_\rho^+} f(z)\mathrm{d}z.$$

Moreover, the second limit on the right-hand side is zero since $f(z)$ is the quotient of two polynomials where the degree of the numerator is 0 and the degree of the denominator is 6, and $6-0 \geq 2$. We are going to calculate the remaining integral over C_ρ by using the residue theorem. The denominator of $f(z)$ can be written $(z-i)^3(z+i)^3$ and, hence, f has two poles (each of order three), namely $-i$ and i, out of which only i lies inside C_ρ (for sufficiently large ρ). The residue of f at i is given by

$$\operatorname{Res}(f;i) = \lim_{z\to i}\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big[(z-i)^3 f(z)\Big] = \lim_{z\to i}\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[\frac{1}{(z+i)^3}\right] = \lim_{z\to i}\frac{1}{2}\frac{12}{(z+i)^5} = \frac{3}{16i}$$

(see Theorem 1 on p. 310 in the course book). Thus

$$\int_0^\infty f(x)\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \frac{1}{2}\lim_{\rho\to\infty} 2\pi i\operatorname{Res}(f;i) = \frac{3\pi}{16}.$$

2. (a) Determine the order of the pole of $f(z) = \frac{1}{(\sin z + z)^2}$ at $z = 0$.

(b) Assume that the analytic function $f(z)$ has a pole of order m at the point z_0. Prove that $f'(z)$ has a pole of order $m+1$ at z_0.

Solution: (a) The function $g(z) = (\sin z + z)^2$ has derivatives

$$\begin{aligned} g'(z) &= 2(\sin z + z)(\cos z + 1),\\ g''(z) &= 2(\cos z + 1)^2 + 2(\sin z + z)(-\sin z + 1), \end{aligned}$$

and, hence, $g(0) = 0, g'(0) = 0, g''(0) = 8 \neq 0$. Thus g has a zero of order two at $z = 0$. Consequently, $f = 1/g$ has a pole of order two at $z = 0$.

(b) It is clear that f' is defined and analytic in a punctured neighborhood of z_0 as f is. If f has a pole of order m at z_0 then f has a Laurent expansion

$$f(z) = \sum_{j=-m}^{\infty} a_j (z - z_0)^j$$

around z_0 with $a_{-m} \neq 0$. Differentiating yields

$$f'(z) = \sum_{j=-m}^{\infty} j a_j (z - z_0)^{j-1} = \sum_{j=-m-1}^{\infty} (j+1) a_{j+1} (z - z_0)^j,$$

and the latter is a (the unique) Laurent expansion of f' around z_0. Its coefficient with the smallest index is $-m a_{-m} \neq 0$ corresponding to $(z - z_0)^{-(m+1)}$. Hence, f' has a pole of order $m + 1$ at z_0.

3. Let γ be a directed smooth curve with initial point α and terminal point β. Show that

$$\int_\gamma z \, \mathrm{d}z = \frac{\beta^2 - \alpha^2}{2}.$$

Which result does this yield if γ is a closed curve? Give an alternative explanation for the result for a closed curve.

Solution: As γ is a smooth curve, we may choose a smooth parametrization $z(t)$, $t \in [0, 1]$, such that $z(0) = \alpha$ and $z(1) = \beta$. Then

$$\int_\gamma z \, \mathrm{d}z = \int_0^1 z(t) z'(t) \, \mathrm{d}t = \frac{1}{2} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} z^2(t) \, \mathrm{d}t = \frac{1}{2} z^2(t) \Big|_{t=0}^{1} = \frac{z^2(1) - z^2(0)}{2} = \frac{\beta^2 - \alpha^2}{2}.$$

For a closed curve γ we have $\alpha = \beta$, that is, $\int_\gamma z \, \mathrm{d}z = 0$. This follows also from Cauchy's integral theorem as z is entire (in particular analytic inside and on γ).

4. Calculate all Laurent series expansions of the function $f(z) = \frac{1}{2z^2 + 4z - 6}$ centered at $z_0 = 1$.

Solution: The denominator can be rewritten $2(z-1)(z+3)$. Hence, f has singularities at 1 and -3 and is analytic otherwise. Hence we have two Laurent expansions centered at 1, namely one for $|z - 1| < 4$ and one for $|z - 1| > 4$. In order to compute them we rewrite $f(z)$ in partial fractions,

$$f(z) = \frac{1}{8(z-1)} - \frac{1}{8(z+3)}. \tag{1}$$

Case 1 ($|z - 1| < 4$): Here

$$\frac{1}{8(z+3)} = \frac{1}{8} \frac{1}{4 - (1 - z)} = \frac{1}{32} \frac{1}{1 - \frac{1-z}{4}} = \frac{1}{32} \sum_{k=0}^{\infty} \left(\frac{1 - z}{4} \right)^k = \frac{1}{32} \sum_{k=0}^{\infty} \left(-\frac{1}{4} \right)^k (z - 1)^k$$

as $|\frac{1-z}{4}| < 1$. Thus (1) gives

$$f(z) = \frac{1}{8}(z - 1)^{-1} - \frac{1}{32} \sum_{k=0}^{\infty} \left(-\frac{1}{4} \right)^k (z - 1)^k.$$

Case 2 ($|z - 1| > 4$): Here

$$\frac{1}{8(z+3)} = \frac{1}{8} \frac{1}{z - 1} \frac{1}{1 - \frac{4}{1-z}} = \frac{1}{8} \frac{1}{z - 1} \sum_{k=0}^{\infty} \left(\frac{4}{1 - z} \right)^k = \frac{1}{8} \sum_{k=0}^{\infty} (-4)^k (z - 1)^{-k-1}.$$

This yields

$$f(z) = -\frac{1}{8}\sum_{k=1}^{\infty}(-4)^k(z-1)^{-k-1} = -\frac{1}{8}\sum_{k=0}^{\infty}(-4)^{k+1}(z-1)^{-k}.$$

5. (a) Use Cauchy's integral formula to determine the value of

$$\oint_{|z|=2} \frac{\cos z}{z^2 - 5z + 4}\,\mathrm{d}z.$$

(b) Suppose that f is analytic inside and on the unit circle $|z| = 1$ and satisfies $|f(z)| \leq M$ for all z with $|z| = 1$. Verify that $|f'(i/2)| \leq 4M$ holds.

Solution: (a) The integrand can be written as $g(z)/(z-1)$, where $g(z) = \frac{\cos z}{z-4}$ is analytic inside and on the given contour. Hence

$$\oint_{|z|=2} \frac{\cos z}{z^2 - 5z + 4}\,\mathrm{d}z = \oint_{|z|=2} \frac{g(z)}{z-1}\,\mathrm{d}z = 2\pi i g(1) = -\frac{2}{3}\cos(1)\pi i$$

by Cauchy's formula.

(b) We apply Cauchy's formula for the derivative and obtain

$$|f'(i/2)| = \left| \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{(z - i/2)^2}\,\mathrm{d}z \right| \leq 4\frac{M}{2\pi}2\pi = 4M,$$

where we have used that the length of the contour equals $2\pi i$ and that for $|z| = 1$ we have

$$|z - i/2| \geq |z| - |i/2| = 1 - 1/2 = 1/2.$$

6. Find a conformal mapping of the first quadrant onto itself which maps the point $1 + i$ to the point $2 + i$.

Solution: We can make life easier by dealing with the upper half-plane. The mapping $f(z) = z^2$ maps the first quadrant onto the upper halfplane, and it maps $1 + i$ to $2i$ and $2 + i$ to $3 + 4i$. A conformal mapping of the upper half-plane onto itself that maps $2i$ onto $3 + 4i$ is given by $g(z) = 2z + 3$. Now a conformal mapping of the first quadrant onto itself with the desired properties is given by

$$\Phi = f^{-1} \circ g \circ f.$$

It is explicitly given by

$$\Phi(z) = \left(2z^2 + 3\right)^{1/2},$$

where the complex square root can be chosen analytic on $\mathbb{C} \setminus (-\infty, 0]$.