MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Jonathan Rohleder

Tentamensskrivning i
Matematik III Komplex Analys
7.5 hp
3 June 2019

No calculators, books, or other resources allowed. Max score on each problem is 5p; grade of E guaranteed at 15p. Appropriate amount of details required for full marks.

1. Find all solutions to the equation $3\sin z + i\cos z = e^{iz}$.

2. Calculate all Laurent series expansions of the function

$$f(z) = \frac{(z-i)^2}{z^2 - (8+i)z + 8i}$$

   centered at $z_0 = i$.

3. Use residue calculus to determine the value of the integral

$$\int_0^\pi \frac{8}{5 + 2\cos x} \mathrm{d}x.$$

4. (a) Show that the function $A \operatorname{Log}|z| + B$ is harmonic in each domain that does not contain the origin if $A, B \in \mathbb{R}$ are constants.
   (b) Find a pair of complex numbers that are symmetric with respect to both the real axis and the circle $|z - 2i| = 1$.
   (c) Determine a harmonic function in $\{z \in \mathbb{C} : \operatorname{Im} z > 0, |z - 2i| > 1\}$ that is equal to π on the circle $|z - 2i| = 1$ and equals zero on the real axis.

5. Formulate Rouché's theorem and use it to prove that each complex polynomial of degree n has exactly n zeroes (taking multiplicities into account).

6. Let

$$G := \{(z, w) \in \mathbb{C}^2 : |w| > 1\}$$

   and let $f : G \to \mathbb{C}$ be analytic and bounded.
   (a) Show that there exists an analytic function g such that $f(z, w) = g(w)$ holds for all $(z, w) \in G$, that is, f is independent of the variable z.
   (b) Show with the help of an appropriate example that a function $f : G \to \mathbb{C}$ which is analytic and bounded is not necessarily constant.
   (c) Is the statement (a) still true if G is replaced by $G' := \{(z, w) \in G : \operatorname{Im} z \neq 0\}$?

Exams will be returned on 5 June 2019 at 3 pm in room 414, building 6, and will be stored in the students' office afterwards.