

No calculators, books, or other resources allowed. Max score on each problem is 5p; grade of E guaranteed at 15p. Appropriate amount of details required for full marks.

1. Use residue calculus to determine the value of the integral

$$\int_0^{\infty} \frac{x^2}{(x^2 + 1)(x^2 + 4)} dx.$$

2. Verify that the function $u(x, y) = 2xy - 5x - x^2 + y^2$ is harmonic and determine all its harmonic conjugates.
3. Calculate all Laurent series expansions of the function

$$f(z) = \frac{1}{(z - 1)^2(z - 2)}$$

centered at $z_0 = 1$.

4. Find the number of zeroes of the function $5z^3 + 9z^2 - 25z + 21$ inside the disc $|z - 1| < 1$.
5. (a) Determine all Möbius transformations that map each pair of parallel straight lines to a pair of parallel straight lines.
- (b) What does the image of a rectangle under a Möbius transformation with the property in (a) look like?
- (c) Find the Möbius transformation that maps 0 to 0, 1 to i and ∞ to ∞ .
6. Let $B = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 \leq 1\}$ denote the closed ball in \mathbb{C}^2 of radius 1 centered at the origin. Assume that $f : \mathbb{C}^2 \setminus B \rightarrow \mathbb{C}$ is analytic and bounded. Show that f is constant on $\mathbb{C}^2 \setminus B$.

Exams will be returned on 28 August 2019 at 3 pm in room 414, building 6, and will be stored in the students' office afterwards.