MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Jonathan Rohleder

Tentamensskrivning i
Matematik III Komplex Analys
7.5 hp
3 June 2019

No calculators, books, or other resources allowed. Max score on each problem is 5p; grade of E guaranteed at 15p. Appropriate amount of details required for full marks.

1. Find all solutions to the equation $3\sin z + i\cos z = e^{iz}$.

   Solution: Using the identities $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$ and $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ and multiplying by i the equation can be rewritten as
   $$e^{iz} - 2e^{-iz} = ie^{iz}.$$
   Multiplying this by e^{iz} and simplifying further gives the equivalent equation
   $$e^{2iz} = 1 + i.$$
   Writing $z = x + iy$ with $x, y \in \mathbb{R}$ we further obtain
   $$e^{2ix}e^{-2y} = 1 + i,$$
   which gives rise to the two equations
   $$e^{-2y} = |1 + i| = \sqrt{2} \quad \text{and} \quad e^{2ix} = \arg(1 + i) = \frac{\pi}{4}$$
   for the real and imaginary part of z. Their solutions are
   $$y = -\frac{\log\sqrt{2}}{2} \quad \text{and} \quad x = \frac{\pi}{8} + k\pi, \quad k \in \mathbb{Z}.$$
   Hence all solutions of the equation $3\sin z + i\cos z = e^{iz}$ are given by
   $$z = \frac{\pi}{8} + k\pi - i\frac{\log\sqrt{2}}{2}, \quad k \in \mathbb{Z}.$$

2. Calculate all Laurent series expansions of the function
   $$f(z) = \frac{(z - i)^2}{z^2 - (8 + i)z + 8i}$$
   centered at $z_0 = i$.

   Solution: Note that f simplifies to
   $$f(z) = \frac{z - i}{z - 8}.$$
   Hence f has a Taylor expansion on $|z - i| < |i - 8| = \sqrt{65}$ given by
   $$f(z) = \frac{z - i}{i - 8} \cdot \frac{1}{1 - \frac{z-i}{8-i}} = -\sum_{j=0}^{\infty} \frac{(z - i)^{j+1}}{(8 - i)^{j+1}}.$$
   On the other hand, for $|z - i| > |i - 8|$ we have a Laurent expansion of the form
   $$f(z) = \frac{1}{1 - \frac{8-i}{z-i}} = \sum_{j=0}^{\infty} \frac{(8 - i)^j}{(z - i)^j}.$$

3. Use residue calculus to determine the value of the integral

$$\int_0^\pi \frac{8}{5+2\cos x}\mathrm{d}x.$$

Solution: We use the expression $2\cos x = e^{ix}+e^{-ix}$ and substitute $z=e^{ix}$ in it. This and the fact that the integrand is symmetric with respect to $x=\pi$ gives

$$\int_0^\pi \frac{8}{5+2\cos x}\mathrm{d}x = \frac{1}{2}\int_0^{2\pi}\frac{8}{5+e^{ix}+e^{-ix}}\mathrm{d}x = \frac{1}{2}\int_C \frac{8}{5+z+\frac{1}{z}}\frac{1}{iz}\mathrm{d}z = \frac{1}{2}\int_C \frac{-8i}{z^2+5z+1}\mathrm{d}z,$$

where C denotes the positively oriented unit circle centered at zero. We are going to solve this integral by means of the residue theorem. For this we look for the singularities of the integrand $f(z)=\frac{-8i}{z^2+5z+1}$ inside the unit circle centered at zero. They must belong to the zeroes of the polynomial z^2+5z+1, and those are given by

$$z=-\frac{5}{2}\pm\sqrt{\frac{25}{4}-1}=\begin{cases}\frac{\sqrt{21}-5}{2},\\ -\frac{\sqrt{21}+5}{2},\end{cases}$$

and precisely one of them, $z_0=\frac{\sqrt{21}-5}{2}$, lies inside the unit circle. It is a pole of order one of the integrand f and we have

$$\operatorname{Res}(f,z_0)=\lim_{z\to z_0}(z-z_0)f(z)=\lim_{z\to\frac{\sqrt{21}-5}{2}}\frac{-8i}{z+\frac{\sqrt{21}+5}{2}}=-\frac{8i}{\sqrt{21}}.$$

Hence the value of the desired integral is

$$\int_0^\pi \frac{8}{5+2\cos x}\mathrm{d}x=\frac{1}{2}2\pi i\operatorname{Res}(f,z_0)=\frac{8\pi}{\sqrt{21}}.$$

4. (a) Show that the function $A\operatorname{Log}|z|+B$ is harmonic in each domain that does not contain the origin if $A,B\in\mathbb{R}$ are constants.
   (b) Find a pair of complex numbers that are symmetric with respect to both the real axis and the circle $|z-2i|=1$.
   (c) Determine a harmonic function in $\{z\in\mathbb{C}:\operatorname{Im}z>0,|z-2i|>1\}$ that is equal to π on the circle $|z-2i|=1$ and equals zero on the real axis.

Solution: (a) Let $D\subset\mathbb{C}$ be a domain that does not contain the origin. Then on the intersection of D with any sector $\{re^{i\phi}:r>0,\phi\in[\phi_1,\phi_2]\}$ with $|\phi_1-\phi_2|<2\pi$ the function $A\operatorname{Log}|z|+B$ is the real part of $A\operatorname{Log}|z|+B+i\arg z$ for an appropriate branch of arg that makes the latter function analytic. Thus $A\operatorname{Log}|z|+B$ is harmonic on this intersection. As the sector can be chosen arbitrarily, it follows that f is analytic on D.

(b) To be symmetric with respect to the real axis, the two points must have the form z, $\overline{z}$. For the symmetry with respect to the circle we further require

$$(z-2i)\overline{(\overline{z}-2i)}=1,$$

which has the solutions $z=\pm\sqrt{3}i$.

(c) The Möbius transformation $f(z)=\frac{z-\sqrt{3}i}{z+\sqrt{3}i}$ maps $\sqrt{3}i$ to 0 and $-\sqrt{3}i$ to ∞. Since every Möbius transformation preserves symmetry w.r.t. circles and lines, these points will still be symmetric to the image circles so that these circles are concentric with center in the origin. The radii of the image circles are

$$|f(i)| = \left|\frac{i-\sqrt{3}i}{i+\sqrt{3}i}\right| = \frac{|1-\sqrt{3}|}{|1+\sqrt{3}|} \qquad \text{and} \qquad |f(0)| = 1.$$

The Dirichlet problem on the domain D' lying between the image circles can be solved using the function in (a). We want to satisfy the boundary values 0 on the circle of radius 1 and π on the circle of radius $\frac{\sqrt{3}-1}{\sqrt{3}+1}$. This implies

$$U(w) = \frac{\pi}{\operatorname{Log}\frac{\sqrt{3}-1}{\sqrt{3}+1}} \operatorname{Log}|w|$$

on D'. Then

$$u(z) = (U \circ f)(z) = \frac{\pi}{\operatorname{Log}\frac{\sqrt{3}-1}{\sqrt{3}+1}} \operatorname{Log}\left|\frac{z-\sqrt{3}i}{z+\sqrt{3}i}\right|$$

is a harmonic function on D with the desired boundary values.

5. Formulate Rouché's theorem and use it to prove that each complex polynomial of degree n has exactly n zeroes (taking multiplicities into account).

   Solution: See Theorem 4 and Example 4 in Section 6.7 in the book by Saff/Snider.

6. Let

$$G := \{(z,w) \in \mathbb{C}^2 : |w| > 1\}$$

   and let $f : G \to \mathbb{C}$ be analytic and bounded.

   (a) Show that there exists an analytic function g such that $f(z,w) = g(w)$ holds for all $(z,w) \in G$, that is, f is independent of the variable z.

   (b) Show with the help of an appropriate example that a function $f : G \to \mathbb{C}$ which is analytic and bounded is not necessarily constant.

   (c) Is the statement (a) still true if G is replaced by $G' := \{(z,w) \in G : \operatorname{Im} z \neq 0\}$?

   Solution: (a) If we fix $w \in \mathbb{C}$ with $|w| > 1$ then the function $z \mapsto f(z,w)$ is analytic on all of $\mathbb{C}$ and bounded and thus by Liouville's theorem there exists a constant $g(w) \in \mathbb{C}$ with $f(z,w) = g(w)$ for all $z \in \mathbb{C}$.

   (b) The function $f(z,w) = \frac{1}{w}$ satisfies $|f(z,w)| = |\frac{1}{w}| < 1$ for $(z,w) \in G$, that is, f is bounded on G. Moreover, f is analytic there, but obviously f is not constant.

   (c) The statement is wrong as the intersection of G' with $w =$ const is not connected any more. For instance the function f given by $f(z,w) = 0$ on $\{(z,w) : \operatorname{Im} z > 0, |w| > 1\}$ and $f(z,w) = 1$ on $\{(z,w) : \operatorname{Im} z < 0, |w| > 1\}$ satisfies all assumptions.

Exams will be returned on 5 June 2019 at 3 pm in room 414, building 6, and will be stored in the students' office afterwards.