MATEMATISKA INSTITUTIONEN	Tentamensskrivning i
STOCKHOLMS UNIVERSITET	Matematik III Komplex Analys
Avd. Matematik	7.5 hp
Examinator: Jonathan Rohleder	23 August 2019

No calculators, books, or other resources allowed. Max score on each problem is 5p; grade of E guaranteed at 15p. Appropriate amount of details required for full marks.

1. Use residue calculus to determine the value of the integral

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\mathrm{d}x.$$

Solution. First note that the integrand is an even function. Hence,

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{(x^2+1)(x^2+4)}.$$

We rewrite the latter integral as the contour integral of the function $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$ along a counterclockwise parametrization of the contour $C_\rho = [-\rho,\rho] \cup C_\rho^+$, where C_ρ^+ denotes the upper half of the circle of radius ρ centered at zero. Thus

$$\int_{-\infty}^\infty \frac{z^2}{(z^2+1)(z^2+4)}\mathrm{d}x = \lim_{\rho\to\infty}\int_{C_\rho} f(z)\mathrm{d}z - \lim_{\rho\to\infty}\int_{C_\rho^+} f(z)\mathrm{d}z.$$

Moreover, the second limit on the right-hand side is zero since $f(z)$ is the quotient of two polynomials where the degree of the numerator is 2 and the degree of the denominator is 4, and $4-2\geq 2$. We are going to calculate the remaining integral over C_ρ by using the residue theorem. One sees directly that f has the four poles (each of order one) $-i$, i, $-2i, 2i$, out of which only i and $2i$ lie inside C_ρ (for sufficiently large ρ). The residue of f at i is given by

$$\mathrm{Res}(f;i) = \lim_{z\to i}\frac{z^2}{(z+i)(z^2+4)} = \frac{i}{6};$$

the residue of f at $2i$ is given by

$$\mathrm{Res}(f;i) = \lim_{z\to 2i}\frac{z^2}{(z^2+1)(z+2i)} = -\frac{i}{3}.$$

Thus by the residue theorem

$$\int_0^\infty f(x)\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \frac{1}{2}\lim_{\rho\to\infty} 2\pi i\big(\mathrm{Res}(f;i)+\mathrm{Res}(f;2i)\big) = \frac{\pi}{6}.$$

2. Verify that the function $u(x,y) = 2xy - 5x - x^2 + y^2$ is harmonic and determine all its harmonic conjugates.

Solution. Computation yields

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2+2 = 0.$$

Hence u is harmonic on $\mathbb{R}^2$. By the Cauchy–Riemann equations, any harmonic conjugate v must satisfy

$$\frac{\partial v}{\partial y}(x,y) = \frac{\partial u}{\partial x}(x,y) = 2y - 5 - 2x,$$

which implies $v(x,y) = y^2 - 5y - 2xy + C(x)$, and

$$\frac{\partial v}{\partial x}(x,y) = -\frac{\partial u}{\partial y} = -2x - 2y.$$

As we already know that $\frac{\partial v}{\partial x}(x,y) = -2y + C'(x)$, we conclude $C'(x) = -2x$ and thus $C(x) = -x^2 + d$, where d is an arbitrary real constant. We conclude that all harmonic conjugates of u have the form

$$v(x,y) = y^2 - 5y - 2xy - x^2 + d$$

with arbitrary $d \in \mathbb{R}$.

3. Calculate all Laurent series expansions of the function

$$f(z) = \frac{1}{(z-1)^2(z-2)}$$

centered at $z_0 = 1$.

Solution. The function f has a pole of order two at 1 and a pole of order 1 at 2. We write f as partial fractions and obtain

$$f(z) = -\frac{1}{z-1} - \frac{1}{(z-1)^2} + \frac{1}{z-2}.$$

For $|z-1| < 1$, $z \neq 1$, we get by using the geometric series

$$f(z) = -\frac{1}{z-1} - \frac{1}{(z-1)^2} - \frac{1}{1-(z-1)} = -\frac{1}{z-1} - \frac{1}{(z-1)^2} - \sum_{j=0}^{\infty}(z-1)^j = -\sum_{j=-2}^{\infty}(z-1)^j.$$

For $|z-1| > 1$ the same argument gives

$$f(z) = -\frac{1}{z-1} - \frac{1}{(z-1)^2} + \frac{1}{z-1}\frac{1}{1-\frac{1}{z-1}} = -\frac{1}{z-1} - \frac{1}{(z-1)^2} + \sum_{j=0}^{\infty}(z-1)^{-j-1} = \sum_{j=3}^{\infty}(z-1)^{-j}.$$

4. Find the number of zeroes of the function $5z^3 + 9z^2 - 25z + 21$ inside the disc $|z-1| < 1$.

Solution. By means of polynomial division we rewrite the given function as

$$5z^3 + 9z^2 - 25z + 21 = 5(z-1)^3 + 24(z-1)^2 + 8(z-1) + 10.$$

Thus our problem is equivalent to finding the number of zeroes of $5w^3 + 24w^2 + 8w + 10$ within $|w| < 1$. We define $f(w) = 24w^2$ and $h(w) = 5w^3 + 8w + 10$. For all w with $|w| = 1$ we have

$$|h(w)| \leq 5|w|^3 + 8|w| + 10 = 23 < 24 = |f(w)|. \tag{1}$$

Moreover, f has two zeroes inside the unit disc (namely a double zero at 0). By Rouché's theorem and (1) also $f(w) + h(w) = 5w^3 + 24w^2 + 8w + 10$ has 2 zeroes inside the unit disc.

5. (a) Determine all Möbius transformations that map each pair of parallel straight lines to a pair of parallel straight lines.

(b) What does the image of a rectangle under a Möbius transformation with the property in (a) look like?

(c) Find the Möbius transformation that maps 0 to 0, 1 to i and ∞ to ∞.

Solution. (a) In the extended complex plane $\hat{\mathbb{C}}$ two lines are parallel if and only if they intersect at ∞. Hence we are looking for all Möbius transformations that map ∞ to ∞. But any transformation of the form $\frac{Az+B}{Cz+D}$ maps ∞ to $\frac{A}{C}$ and thus the requirement is equivalent to $C = 0$. Thus with $a = A/D$ and $b = B/D$ the Möbius transformations in question are those that have the form $az + b$, i.e. the affine-linear mappings.

(b) As the transformations under consideration map parallel lines onto parallel lines and preserve angles, the image of any rectangle will again be a rectangle.

(c) As the Möbius transformation we are looking for shall map ∞ to ∞, it belongs to the above class and takes the form $f(z) = az + b$ with complex a, b. Moreover, the requirement $f(0) = 0$ enforces $b = 0$. Finally, from $f(1) = i$ we get $a = i$. Thus $f(z) = iz$.

6. Let $B = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 \leq 1\}$ denote the closed ball in $\mathbb{C}^2$ of radius 1 centered at the origin. Assume that $f : \mathbb{C}^2 \setminus B \to \mathbb{C}$ is analytic and bounded. Show that f is constant on $\mathbb{C}^2 \setminus B$.

Solution. For each $w_0 \in \mathbb{C}$ with $|w_0| > 1$, any point (z, w_0) with $z \in \mathbb{C}$ satisfies $|(z, w_0)|^2 = |z|^2 + |w|^2 > 1$ and thus the function $z \mapsto f(z, w_0)$ is entire and, by assumption, bounded. Thus by Liouville's theorem, for each $|w_0| > 1$ this function is constant. Similarly, for each $|z_0| > 1$ the function $w \mapsto f(z_0, w)$ is constant. It follows that f is constant on the set

$$\big\{(z, w) \in \mathbb{C}^2 : |z| > 1 \text{ or } |w| > 1\big\} \subsetneq \mathbb{C}^2 \setminus B.$$

Take now again the function $z \mapsto f(z, w_0)$, but for arbitrary $w_0 \in \mathbb{C}$; it is defined on all z such that (z, w_0) is outside B and it is analytic there. On the other hand, it is constant for all sufficiently large z and, as an analytic function, must then be constant on its whole (connected) domain (since it is constant on a set with accumulation point). This implies that f is constant everywhere outside B.

Exams will be returned on 28 August 2019 at 3 pm in room 414, building 6, and will be stored in the students' office afterwards.