MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Jonathan Rohleder

Tentamensskrivning i
Matematik III Komplex Analys
7.5 hp
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No calculators, books, or other resources allowed. Max score on each problem is 5p; grade of E guaranteed at 15p. Appropriate amount of details required for full marks.

1. Calculate all Laurent series expansions of the function

$$f(z) = \frac{1}{z^2 + (4-3i)z - 12i}$$

centered at $z_0 = 3i$.

Solution: f is a rational function with poles at the zeroes of the denominator, which are $3i$ and -4. The zero at $3i$ equals z_0, the distance of the other one to -4 is

$$|3i + 4| = \sqrt{9 + 16} = 5.$$

Hence there are two Laurent expansions of f centered at z_0, namely one for $0 < |z - z_0| < 5$ and one for $|z - z_0| > 5$. We rewrite f as

$$f(z) = \frac{1}{(z-3i)(z+4)} = \frac{\alpha}{z-3i} - \frac{\alpha}{z+4}, \qquad \text{where} \qquad \alpha = \frac{4}{25} - \frac{3i}{25}$$

(partial fractions). For $0 < |z - z_0| < 5$ we get $|\frac{z-z_0}{z_0+4}| < 1$ and hence

$$\frac{1}{z+4} = \frac{1}{(z_0+4)\big(1 - \big(-\frac{z-z_0}{z_0+4}\big)\big)} = \frac{1}{z_0+4}\sum_{k=0}^{\infty}(-1)^k \frac{(z-z_0)^k}{(z_0+4)^k}.$$

Hence, for $0 < |z - z_0| < 5$ we get

$$f(z) = \alpha\left(\frac{1}{z-3i} - \sum_{k=0}^{\infty}(-1)^k \frac{(z-3i)^k}{(3i+4)^{k+1}}\right).$$

Moreover, for $|z - z_0| > 5$ we have $|\frac{z_0+4}{z-z_0}| < 1$ and, hence,

$$\begin{aligned}\frac{1}{z+4} &= \frac{1}{(z-z_0)\big(1 - \big(-\frac{z_0+4}{z-z_0}\big)\big)} = (z-z_0)^{-1}\sum_{k=0}^{\infty}(-1)^k (z_0+4)^k (z-z_0)^{-k} \\ &= \sum_{k=0}^{\infty}(-1)^k (z_0+4)^k (z-z_0)^{-k-1}.\end{aligned}$$

Thus for $|z - z_0| > 5$ we get

$$f(z) = \alpha\sum_{k=2}^{\infty}(-1)^k (z_0+4)^{k-1}(z-z_0)^{-k}.$$

2. Find all $a, b, c \in \mathbb{R}$ such that $ax^2 + be^{x-y} + cy^2$ is the real part of an analytic function. Moreover, for each such triple (a, b, c) determine all these analytic functions.

Solution: Let $u(x,y) = ax^2 + be^{x-y} + cy^2$. The real part of an analytic function is always harmonic. Hence we are looking for a, b, c such that $0 = \Delta u(x,y) = 2a + be^{x-y} + be^{x-y} + 2c$, which is equivalent to $b = 0$ and $c = -a$. Thus we are looking for a harmonic conjugate to the function $u(x,y) = a(x^2 - y^2)$, where $a \in \mathbb{R}$. In order to find such a function, note that

$$\frac{\partial u}{\partial x}(x,y) = 2ax, \qquad \frac{\partial u}{\partial y}(x,y) = -2ay.$$

According to the Cauchy–Riemann differential equations, any harmonic conjugate v of u must satisfy

$$\frac{\partial v}{\partial y}(x,y) = \frac{\partial u}{\partial x}(x,y) = 2ax,$$

and integration implies $v(x,y) = 2axy + C(x)$ with a function C depending only on x. Using the second Cauchy–Riemann equation we find

$$2ay + C'(x) = \frac{\partial v}{\partial x}(x,y) = -\frac{\partial u}{\partial y}(x,y) = 2ay.$$

Hence, $C'(x) = 0$, which implies $C(x) = D$ for a constant $D \in \mathbb{R}$. Thus each analytic function having u as its real part has the form

$$a(x^2 - y^2) + i\big(2axy + D\big), \qquad D \in \mathbb{R}.$$

3. Determine the value of the integral

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)}\,\mathrm{d}x,$$

where $a, b > 0$ are real with $a \neq b$.

Solution: We rewrite the integral as the contour integral of the function $f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$ along a counterclockwise parametrization of the contour $C_\rho = [-\rho, \rho] \cup C_\rho^+$, where C_ρ^+ denotes the upper half of the circle of radius ρ centered at zero. Thus

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)}\,\mathrm{d}x = \lim_{\rho\to\infty} \int_{C_\rho} f(z)\mathrm{d}z - \lim_{\rho\to\infty} \int_{C_\rho^+} f(z)\mathrm{d}z.$$

Moreover, the second limit on the right-hand side is zero since $f(z)$ is the quotient of two polynomials where the degree of the numerator is 0 and the degree of the denominator is 4, and $4 - 0 \geq 2$. We are going to calculate the remaining integral over C_ρ by using the residue theorem. The singularities of the function f are $\pm ai$ and $\pm bi$. None of them lie on the real axis, but two of them lie in the upper complex half-plane, namely $z_1 = ai$ and $z_2 = bi$. As these are simple zeroes of the denominator, we may calculate the residues of f at these points,

$$\mathrm{Res}(f; z_1) = \lim_{z\to ai}(z - ai)f(z) = \lim_{z\to ai} \frac{1}{(z+ai)(z^2+b^2)} = \frac{1}{2ai(b^2-a^2)}$$

and analogously

$$\mathrm{Res}(f; z_2) = \frac{1}{2bi(a^2-b^2)}.$$

Thus

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{\rho\to\infty} 2\pi i \sum_{j=1}^{2} \mathrm{Res}(f; z_j) = \frac{\pi}{b^2-a^2}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{\pi}{ab(a+b)}.$$

4. Let $a \in \mathbb{C}$ with $|a| > e$. Prove that the equation

$$e^z = az^n$$

has precisely n solutions in the open unit disc $|z| < 1$.

Solution: We trace the problem back to Rouché's theorem. Define $f(z) = -z^n$ and $g(z) = \frac{e^z}{a} - z^n$. (Note that $a \neq 0$ by assumption.) Then the equation $e^z = az^n$ is equivalent to $g(z) = 0$. Both functions g and f are analytic inside and on the contour $|z| = 1$. Moreover, if we are on the contour, i.e. $|z| = 1$, we have

$$|g(z) - f(z)| = \frac{e^{\operatorname{Re} z}}{|a|} < \frac{e}{e} = 1 = |-z^n| = |f(z)|.$$

Consequently, g has the same number of zeroes inside the contour as f, namely n.

5. Show that if f is analytic at z_0 and $f'(z_0) \neq 0$ then there exists an open disk D centered at z_0 such that f is injective on D. (In particular, f is conformal on D.)

Solution: See Theorem 1 in Section 7.2 in the book by Saff/Snider.

6. (a) Show that the function $A \operatorname{Log}|z| + B$, with $A, B \in \mathbb{R}$ constant, is harmonic in each domain that does not contain the origin.

(b) Find a pair of complex numbers that are symmetric with respect to both the real axis and the circle $|z + 5i| = 4$.

(c) Determine a harmonic function in $\{z \in \mathbb{C} : \operatorname{Im} z < 0, |z + 5i| > 4\}$ that is equal to 0 on the circle $|z + 5i| = 4$ and equals 1 on the real axis.

Solution: (a) Let $D \subset \mathbb{C}$ be a domain that does not contain the origin. Then on the intersection of D with any sector $\{re^{i\phi} : r > 0, \phi \in [\phi_1, \phi_2]\}$ with $|\phi_1 - \phi_2| < 2\pi$ the function $A \operatorname{Log}|z| + B$ is the real part of $A \operatorname{Log}|z| + B + i \arg z$ for an appropriate branch of arg that makes the latter function analytic. Thus $A \operatorname{Log}|z| + B$ is harmonic on this intersection. As the sector can be chosen arbitrarily, it follows that f is analytic on D.

(b) To be symmetric with respect to the real axis, the two points must have the form z, $\overline{z}$. For the symmetry with respect to the circle we further require

$$(z + 5i)\overline{(\overline{z} + 5i)} = 16,$$

which has the solutions $z = \pm 3i$.

(c) The Möbius transformation $f(z) = \frac{z-3i}{z+3i}$ maps $3i$ to 0 and $-3i$ to ∞. Since every Möbius transformation preserves symmetry w.r.t. circles and lines, these points will still be symmetric to the image circles so that these circles are concentric with center in the origin. The radii of the image circles are

$$|f(-i)| = \left|\frac{-4i}{2i}\right| = 2 \qquad \text{and} \qquad |f(0)| = 1.$$

The Dirichlet problem on the domain D' lying between the image circles can be solved using the function in (a). We want to satisfy the boundary values 0 on the circle of radius 2 and 1 on the circle of radius 1 centered at the origin. This implies

$$U(w) = -\frac{1}{\operatorname{Log} 2} \operatorname{Log}|w| + 1$$

on D'. Then

$$u(z) = (U \circ f)(z) = -\frac{1}{\operatorname{Log} 2} \operatorname{Log}\left|\frac{z - 3i}{z + 3i}\right| + 1$$

is a harmonic function on D with the desired boundary values.