MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Tentamensskrivning i
Matematik III Komplex Analys
7.5 hp
15 January 2020

No calculators, books, or other resources allowed. Max score on each problem is 5p; grade of E guaranteed at 15p. Appropriate amount of details required for full marks.

1. Calculate all Laurent series expansions of the function

$$f(z) = \frac{1}{z^2 - (4+i)z + 3 + 3i}$$

centered at $z_0 = 0$.

Solution: Rewrite f as $f(z) = \frac{1}{(z-3)(z-(1+i))}$. Its singularities are simple poles at 3 and $1+i$, so that we get three Laurent series expansions: a Taylor expansion for $|z| < |1+i| = \sqrt{2}$ and proper Laurent expansions for $\sqrt{2} < |z| < 3$ and $|z| > 3$. In any case we use partial fractions to write

$$f(z) = \frac{2+i}{5}\left(\frac{1}{z-3} - \frac{1}{z-(1+i)}\right).$$

$|z| < \sqrt{2}$: We have

$$f(z) = \frac{2+i}{5}\left(-\frac{1}{3}\frac{1}{1-\frac{z}{3}} + \frac{1}{1+i}\frac{1}{1-\frac{z}{1+i}}\right) = \frac{2+i}{5}\left(-\frac{1}{3}\sum_{j=0}^{\infty}\frac{z^j}{3^j} + \frac{1}{1+i}\sum_{j=0}^{\infty}\frac{z^j}{(1+i)^j}\right).$$

$\sqrt{2} < |z| < 3$: We have $|\frac{1+i}{z}| < 1$ and, hence,

$$f(z) = \frac{2+i}{5}\left(-\frac{1}{3}\frac{1}{1-\frac{z}{3}} - \frac{1}{z}\frac{1}{1-\frac{1+i}{z}}\right) = \frac{2+i}{5}\left(-\frac{1}{3}\sum_{j=0}^{\infty}\frac{z^j}{3^j} - \sum_{j=0}^{\infty}\frac{(1+i)^j}{z^{j+1}}\right).$$

$|z| > 3$: Now $|\frac{1+i}{z}| < 1$ and $|\frac{3}{z}| < 1$ and, hence,

$$f(z) = \frac{2+i}{5}\left(\frac{1}{z}\frac{1}{1-\frac{3}{z}} - \frac{1}{z}\frac{1}{1-\frac{1+i}{z}}\right) = \frac{2+i}{5}\left(\sum_{j=0}^{\infty}\frac{3^j}{z^{j+1}} - \sum_{j=0}^{\infty}\frac{(1+i)^j}{z^{j+1}}\right).$$

2. Compute the contour integral

$$\int_\Gamma \left(|z-1+i|^2 - z\right) \mathrm{d}z,$$

where Γ is the positively oriented upper semicircle of radius one centered at $1-i$.

Solution: We parametrize the contour by $\gamma(t) = 1 - i + e^{it}$, $t \in [0, \pi]$ and have $\gamma'(t) = ie^{it}$. Now the contour integral can be evaluated as

$$\int_\Gamma \left(|z-1+i|^2 - z\right) \mathrm{d}z = \int_0^\pi (i - e^{it}) ie^{it}\,\mathrm{d}t = \cdots = -2i.$$

3. Use residue calculus to determine the value of the integral

$$\int_0^{2\pi} \frac{2}{\cos x + 3}\,\mathrm{d}x.$$

Solution: We use the expression $2\cos x = e^{ix} + e^{-ix}$ and substitute $z = e^{ix}$ in it. This gives

$$\int_0^{2\pi} \frac{2}{\cos x + 3}\,\mathrm{d}x = \int_C \frac{4}{z + \frac{1}{z} + 6}\frac{1}{iz}\mathrm{d}z = \int_C \frac{-4i}{z^2 + 6z + 1}\mathrm{d}z,$$

where C denotes the positively oriented unit circle centered at zero. We are going to solve this integral by means of the residue theorem. For this we look for the singularities of the integrand $f(z) = \frac{-4i}{z^2+6z+1}$ inside the unit circle centered at zero. They must belong to the zeroes of the polynomial $z^2 + 6z + 1$, and those are given by

$$z = -3 \pm \sqrt{8},$$

and precisely one of them, $z_0 = -3 + \sqrt{8}$, lies inside the unit circle. It is a pole of order one of the integrand f and we have

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to -3+\sqrt{8}} \frac{-4i}{z + 3 + \sqrt{8}} = -\frac{2i}{\sqrt{8}}.$$

Hence the value of the desired integral is

$$\int_0^{2\pi} \frac{2}{\cos x + 3}\,\mathrm{d}x = 2\pi i \operatorname{Res}(f, z_0) = \frac{4\pi}{\sqrt{8}} = \sqrt{2}\pi.$$

4. (a) Show that $u(x, y) = \operatorname{Re} f(z)$ is harmonic for any analytic function f, where $z = x + iy$.

(b) Prove that the harmonic conjugate of a given harmonic function u is unique up to a constant: If v_1, v_2 are two harmonic conjugates of u then $v_1 - v_2$ is constant.

Solution: (a) If f is analytic, it is infinitely often (real) differentiable and satisfies the Cauchy–Riemann differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

As a consequence, using the Schwarz lemma we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0.$$

Thus u is harmonic.

(b) If v_1, v_2 are harmonic conjugates to u then the Cauchy–Riemann equations apply to both the functions $u + iv_1$ and $u + iv_2$ and we get

$$\frac{\partial}{\partial y}(v_1 - v_2) = \frac{\partial}{\partial x}(u - u) = 0 \qquad \text{and} \qquad \frac{\partial}{\partial x}(v_1 - v_2) = -\frac{\partial}{\partial y}(u - u) = 0,$$

which implies that $v_1 - v_2$ is constant.

5. Sketch the region

$$S = \left\{ z = re^{i\varphi} : r > 0, 0 < \varphi < \frac{3\pi}{4} \right\}$$

in the complex plane and find a conformal mapping of the open disc of radius one centered at the origin onto S that maps the origin onto the point $e^{\frac{3\pi}{8}i}$.

Solution: The region S is an infinite sector in the complex plane with its vertex sitting at the origin and its angle between 0 and $\frac{3\pi}{4}$.

We compute the desired conformal mapping as a concatenation $f = h \circ g$, where

- g is a Möbius transformation from the unit disc onto the upper halfplane and
- h maps the upper halfplane onto S.

In more detail, we may choose

$$g(z) = i\frac{1-z}{1+z},$$

which maps the unit disc onto the upper halfplane and 0 onto i. Moreover, the mapping h can be chosen as

$$h(w) = w^{\frac{3}{4}}$$

where we choose the branch cut of the power $3/4$ e.g. on the positive real axis. Then, indeed, $h(i) = e^{\frac{3\pi}{8}i}$ and, thus,

$$f(z) = \left(i\frac{1-z}{1+z}\right)^{3/4}$$

has the desired properties.

6. Compute the integral

$$\iint_{\partial_0 P} \frac{3}{1-2zw}\,\mathrm{d}z\,\mathrm{d}w,$$

where $\partial_0 P = \{(z,w) : |z| = |w| = 1\}$ is the distinguished boundary of the unit polydisk centered at the origin, taken with the usual orientation.

Solution: We will twice apply the one-variable version of Cauchy's integral formula. We rewrite the integral as

$$\iint_{\partial_0 P} \frac{3}{1-2zw}\,\mathrm{d}z\,\mathrm{d}w = -\int_{|w|=1} \frac{3}{2w} \int_{|z|=1} \frac{1}{z-\frac{1}{2w}}\,\mathrm{d}z\,\mathrm{d}w,$$

and for w with $|w| = 1$ the inner integral has the form $\int_{|z|=1} \frac{g(z)}{z-z_0}\,\mathrm{d}z$ with z_0 in the interior of the unit disk and $g(z) = 1$. We can apply Cauchy's integral formula to get

$$\iint_{\partial_0 P} \frac{3}{1-2zw}\,\mathrm{d}z\,\mathrm{d}w = -\int_{|w|=1} \frac{3}{2w} 2\pi i\,\mathrm{d}w.$$

Using Cauchy's integral formula once more in an analogous way (to the point $w_0 = 0$) we finally get

$$\iint_{\partial_0 P} \frac{3}{1-2zw}\,\mathrm{d}z\,\mathrm{d}w = -3\pi i 2\pi i = 6\pi^2.$$