MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Tentamensskrivning i
Matematik III Komplex Analys
7.5 hp
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As the exam questions were individualized, we provide solutions to one set of example questions.

1. Find all solutions to the equation $\tan z = \frac{1}{2}(1+\sqrt{3}i)$.

   Solution: The equation can be rewritten
$$-i\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}} = \frac{1}{2}(1+\sqrt{3}i)$$
   and then
$$\frac{e^{2zi}-1}{e^{2zi}+1} = \frac{1}{2}(i-\sqrt{3}).$$
   After multiplying by $e^{2zi}+1$ and simplifying we get
$$e^{2iz} = \frac{i}{2+\sqrt{3}} = e^{-\mathrm{Log}\,(2+\sqrt{3})+\frac{\pi i}{2}}.$$
   This yields
$$z = \frac{\pi}{4} + \frac{i}{2}\mathrm{Log}\,(2+\sqrt{3}) + \pi k, \qquad k \in \mathbb{Z}.$$

2. Calculate all Laurent series expansions of the function
$$f(z) = \frac{1}{(z-1)(z-i)}$$
   centered at $z_0 = 0$.

   Solution: The function f has poles of order one at 1 and i, both at distance 1 from the origin. Hence we have different Laurent series expansions for $|z|<1$ and $|z|>1$. Moreover, we have
$$f(z) = \left(\frac{1}{2}+\frac{i}{2}\right)\left(\frac{1}{z-1} - \frac{1}{z-i}\right).$$
   Hence for $|z|<1$ we have for $a \in \{1, i\}$
$$\frac{1}{z-a} = \frac{1}{-a}\frac{1}{1-\frac{z}{a}} = -\sum_{j=0}^{\infty}\frac{z^j}{a^{j+1}}.$$
   Thus in $|z|<1$ we have
$$f(z) = \left(\frac{1}{2}+\frac{i}{2}\right)\left(-\sum_{j=0}^{\infty} z^j + \sum_{j=0}^{\infty}\frac{z^j}{i^{j+1}}\right) = \left(\frac{1}{2}+\frac{i}{2}\right)\sum_{j=0}^{\infty}(i^{-j-1}-1)z^j.$$
   For $|z|>1$ and $a \in \{1, i\}$ we get
$$\frac{1}{z-a} = \frac{1}{z}\frac{1}{1-\frac{a}{z}} = \sum_{j=0}^{\infty} a^j z^{-j-1}.$$
   Hence in $|z|>1$ we have
$$f(z) = \left(\frac{1}{2}+\frac{i}{2}\right)\sum_{j=0}^{\infty}(1-i^j)z^{-j-1}.$$

3. Use residue calculus to determine the value of the integral

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+9)} \mathrm{d}x.$$

Solution: First note that the integrand is an even function. Hence,

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+9)} \, \mathrm{d}x = \frac{1}{2} \int_{-\infty}^\infty \frac{x^2}{(x^2+1)(x^2+9)}.$$

We rewrite the latter integral as the contour integral of the function $f(z) = \frac{z^2}{(z^2+1)(z^2+9)}$ along a counterclockwise parametrization of the contour $C_\rho = [-\rho, \rho] \cup C_\rho^+$, where C_ρ^+ denotes the upper half of the circle of radius ρ centered at zero. Thus

$$\int_{-\infty}^\infty \frac{z^2}{(z^2+1)(z^2+9)} \mathrm{d}x = \lim_{\rho\to\infty} \int_{C_\rho} f(z) \mathrm{d}z - \lim_{\rho\to\infty} \int_{C_\rho^+} f(z) \mathrm{d}z.$$

Moreover, the second limit on the right-hand side is zero since $f(z)$ is the quotient of two polynomials where the degree of the numerator is 2 and the degree of the denominator is 4, and $4-2 \geq 2$. We are going to calculate the remaining integral over C_ρ by using the residue theorem. One sees directly that f has the four poles (each of order one) $-i$, i, $-3i, 3i$, out of which only i and $3i$ lie inside C_ρ (for sufficiently large ρ). The residue of f at i is given by

$$\operatorname{Res}(f; i) = \lim_{z\to i} \frac{z^2}{(z+i)(z^2+9)} = \frac{i}{16};$$

the residue of f at $3i$ is given by

$$\operatorname{Res}(f; 3i) = \lim_{z\to 3i} \frac{z^2}{(z^2+1)(z+3i)} = -\frac{3i}{16}.$$

Thus by the residue theorem

$$\int_0^\infty f(x) \, \mathrm{d}x = \frac{1}{2} \int_{-\infty}^\infty f(x) \, \mathrm{d}x = \frac{1}{2} \lim_{\rho\to\infty} 2\pi i \big(\operatorname{Res}(f; i) + \operatorname{Res}(f; 3i)\big) = \frac{\pi}{8}.$$

4. Assume that $f(z)$ is analytic in the disk $|z| < 2$ and continuous in the closed disk $|z| \leq 2$ with $|f(z)| \leq 48$ for all z with $|z| = 2$. Moreover, assume that $f(z)/z^3$ is analytic in $|z| < 2$ as well. Find an upper bound for $|f(i/6)|$ and show that it is optimal.

Solution: For $|z| = 2$ we have

$$\left|\frac{f(z)}{z^3}\right| = \frac{|f(z)|}{8} \leq \frac{48}{8} = 6.$$

It follows from the maximum modulus principle that

$$\left|\frac{f(z)}{z^3}\right| \leq 6 \quad \text{for all } z \text{ with } |z| \leq 2$$

and, hence, $|f(z)| \leq 6|z|^3$ for all z with $|z| \leq 2$. Therefore

$$|f(i/6)| \leq 6\frac{1}{6^3} = \frac{1}{36}.$$

This estimate is optimal: The function $f(z) = 6z^3$ is analytic and $f(z)/z^3 = 6$ is analytic as well. Moreover, on $|z| = 2$ we have $|f(z)| = 48$. Finally,

$$|f(i/6)| = 6\frac{1}{6^3} = \frac{1}{36}.$$

5. Prove that all zeroes of the polynomial $z^6 - 5z^2 + 10$ lie in the annulus $1 < |z| < 2$.

Solution: Let $g(z) = z^6 - 5z^2 + 10$ and $f_1(z) = 10$. Then for $|z| = 1$ we have

$$|g(z) - f_1(z)| \le |z|^6 + 5|z|^2 = 6 < 10 = |f(z)|.$$

Hence g has no zeroes on $|z| = 1$ and no zeroes in $|z| < 1$ either. On the other hand, with $f_2(z) = z^6$, on $|z| = 2$ we have

$$|g(z) - f_2(z)| \le 5 \cdot 2^2 + 10 = 30 < 2^6 = |f_2(z)|,$$

and it follows that g has no zeroes on $|z| = 2$ and 6 zeroes in $|z| < 2$. By the above reasoning, all of them have to lie in $1 < |z| < 2$. Moreover, as g is a polynomial of degree 6, no further zeroes exist.

6. (a) Show that, for $A, B \in \mathbb{R}$ constant, the function $A \operatorname{Log}|z| + B$ is harmonic in each domain that does not contain the origin.
   (b) Find a pair of complex numbers that are symmetric with respect to both circles $|z| = 1$ and $|z - \frac{3}{10}| = \frac{3}{10}$.
   (c) Determine a harmonic function in the domain enclosed by the two circles in (b) that is constantly equal to zero on $|z - \frac{3}{10}| = \frac{3}{10}$ and constantly equal to 1 on $|z| = 1$.

Solution: (a) Let $D \subset \mathbb{C}$ be a domain that does not contain the origin. Then on the intersection of D with any sector $\{re^{i\phi} : r > 0, \phi \in [\phi_1, \phi_2]\}$ with $|\phi_1 - \phi_2| < 2\pi$ the function $A \operatorname{Log}|z| + B$ is the real part of $A \operatorname{Log}|z| + B + i \arg z$ for an appropriate branch of arg that makes the latter function analytic. Thus $A \operatorname{Log}|z| + B$ is harmonic on this intersection. As the sector can be chosen arbitrarily, it follows that f is analytic on D.

(b) Symmetry with respect to $|z| = 1$ means

$$z_1 \overline{z_2} = 1,$$

while symmetry with respect to $|z - \frac{3}{10}| = \frac{3}{10}$ can be written as

$$\left(z_1 - \frac{3}{10}\right) \overline{\left(z_2 - \frac{3}{10}\right)} = \frac{9}{100}.$$

The solution to this system of equations is $z_1 = \frac{1}{3}$, $z_2 = 3$.

(c) The Möbius transformation $f(z) = \frac{z - \frac{1}{3}}{z - 3}$ maps $1/3$ to 0 and 3 to ∞. Since every Möbius transformation preserves symmetry w.r.t. circles and lines, these points will still be symmetric to the image circles so that these circles are concentric with center in the origin. The radii of the image circles are

$$|f(1)| = \left| \frac{1 - \frac{1}{3}}{1 - 3} \right| = \frac{1}{3} \qquad \text{and} \qquad |f(0)| = \frac{1}{9}.$$

The Dirichlet problem on the domain D' lying between the image circles can be solved using the function in (a). We want to satisfy the boundary values 1 on the circle of radius $1/3$ and 0 on the circle of radius $1/9$. This implies

$$U(w) = \frac{\operatorname{Log}|w|}{\operatorname{Log} 3} + 2$$

on D'. Then

$$u(z) = (U \circ f)(z) = \frac{\operatorname{Log} \left| \frac{z - \frac{1}{3}}{z - 3} \right|}{\operatorname{Log} 3} + 2$$

is a harmonic function on D with the desired boundary values.