1. (a)
$$\dfrac{\begin{matrix}[\varphi] \\ \vdots \\ \psi\end{matrix}}{\varphi \to \psi} \to I$$

(b)
$$\dfrac{\forall x\, \varphi}{\varphi[t/x]} \forall E$$

with t free for x in φ

(c)
$$\dfrac{\exists x\, \varphi \qquad \left.\begin{matrix}[\varphi] \\ \vdots \\ \sigma\end{matrix}\right\} \mathcal{D}}{\sigma} \exists E$$

with x not free in σ, and x not free in any undischarged assumption of $\mathcal{D}$, except possibly φ

2. Consider $\mathcal{A}$ and $\mathcal{A}'$ given as follows:
$P_1^{\mathcal{A}}$ false, $P_2^{\mathcal{A}}$ true, $P_3^{\mathcal{A}}$ true

$P_1^{\mathcal{A}'}$ false, $P_2^{\mathcal{A}'}$ false, $P_3^{\mathcal{A}'}$ true

We have the following truth table:

	P_1	P_2	P_3	$P_2 \to P_1$	$\neg(P_2 \to P_1)$	$P_2 \wedge P_3$
in $\mathcal{A}$	0	1	1	0	1	1
in $\mathcal{A}'$	0	0	1	1	0	0

Therefore both $\neg(P_2 \to P_1)$ and $P_2 \wedge P_3$ are true in $\mathcal{A}$ and false in $\mathcal{A}'$.

3. (a) When using the rule $\exists E$, x_1 cannot be free in any undischarged assumption of the derivation on the right. But the assumption $\neg P_1(x_1)$ is discharged below $\exists E$, so it is not considered discharged when checking the conditions for $\exists E$. The variable x_1 is free in $\neg P_1(x_1)$, therefore the conditions are not satisfied and

the derivation is incorrect.

(b) If there is a derivation of $\exists x_1\, P_1(x_1) \vdash \neg\neg P_1(x_1)$, then for any interpretation $\mathcal{A}$ and valuation v such that $\llbracket \exists x_1\, P_1(x_1) \rrbracket^{\mathcal{A},v} = 1$, we have $\llbracket \neg\neg P_1(x_1) \rrbracket^{\mathcal{A},v} = 1$, by the soundness theorem.
Take $\mathcal{A}$ such that $|\mathcal{A}| = \{0,1\}$ and $P_1^{\mathcal{A}}$ is false at 0 and true at 1, and take v such that $v(x_1) = 0$.

We have $\llbracket \exists x_1\, P_1(x_1) \rrbracket^{\mathcal{A},v} = 1$ because there is an element of the domain for which $P_1^{\mathcal{A}}$ is true, but $\llbracket \neg\neg P_1(x_1) \rrbracket^{\mathcal{A},v} = 0$ because $P_1^{\mathcal{A}}$ is false at $v(x_1)$.

Therefore there is no derivation of $\exists x_1\, P_1(x_1) \vdash \neg\neg P_1(x_1)$

4. To show that Γ is inconsistent, it is enough to give a derivation of $(P_1 \leftrightarrow \neg P_1) \vdash \bot$. Here is one:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P_1 \leftrightarrow \neg P_1}{P_1 \to \neg P_1}{\scriptstyle \wedge E} \quad [P_1]^1}{\neg P_1}{\scriptstyle \to E} \quad [P_1]^1}{\bot}{\scriptstyle \to E}}{\neg P_1}{\scriptstyle \to I,1} \quad \dfrac{\dfrac{P_1 \leftrightarrow \neg P_1}{\neg P_1 \to P_1}{\scriptstyle \wedge E} \quad \dfrac{\dfrac{\dfrac{\dfrac{P_1 \leftrightarrow \neg P_1}{P_1 \to \neg P_1}{\scriptstyle \wedge E} \quad [P_1]^2}{\neg P_1}{\scriptstyle \to E} \quad [P_1]^2}{\bot}{\scriptstyle \to E}}{\neg P_1}{\scriptstyle \to I,2}}{P_1}{\scriptstyle \to E}}{\bot}{\scriptstyle \to E}$$

To show that Γ_1 and Γ_2 are consistent, it is enough to construct a model for each of them, by the soundness theorem. But $P_1 \to \neg P_1$ is true as soon as P_1 is false, and $\neg P_1 \to P_1$ is true as soon as P_1 is true, so $\mathcal{A}_1$ such that $P_1^{\mathcal{A}_1}$ is false is a model of Γ_1 and $\mathcal{A}_2$ such that $P_1^{\mathcal{A}_2}$ is true in a model of Γ_2.

5. (a) x_1, x_2 and x_3

(b) x_2

6. (a)

$$\dfrac{\dfrac{\dfrac{[\neg\neg P_1]^1 \quad [\neg P_1]^2}{\bot}\,{\scriptstyle \to E}}{P_1}\,{\scriptstyle RAA,2}}{\neg\neg P_1 \to P_1}\,{\scriptstyle \to I,1}$$

(b)

$$\dfrac{\dfrac{\dfrac{\dfrac{P_1 \to (P_2 \wedge P_3) \quad [P_1]^1}{P_2 \wedge P_3}\,{\scriptstyle \to E}}{P_2}\,{\scriptstyle \wedge E}}{P_1 \to P_2}\,{\scriptstyle \to I,1} \qquad \dfrac{\dfrac{\dfrac{P_1 \to (P_2 \wedge P_3) \quad [P_1]^2}{P_2 \wedge P_3}\,{\scriptstyle \to E}}{P_3}\,{\scriptstyle \wedge E}}{P_1 \to P_3}\,{\scriptstyle \to I,2}}{(P_1 \to P_2) \wedge (P_1 \to P_3)}\,{\scriptstyle \wedge I}$$

(c)

$$\dfrac{\dfrac{\dfrac{[\varphi]^1}{\varphi \vee \varphi}\,{\scriptstyle \vee I}}{\varphi \to \varphi \vee \varphi}\,{\scriptstyle \to I,1} \qquad \dfrac{\dfrac{[\varphi \vee \varphi]^2 \quad [\varphi]^3 \quad [\varphi]^3}{\varphi}\,{\scriptstyle \vee E,3}}{\varphi \vee \varphi \to \varphi}\,{\scriptstyle \to I,2}}{\varphi \leftrightarrow \varphi \vee \varphi}\,{\scriptstyle \wedge I}$$

7. (a) Consider a derivation ending with the $\top I$ rule:

$$\mathcal{D}:\ \dfrac{}{\top}\,{\scriptstyle \top I}$$

We have to show that if all undischarged assumptions of $\mathcal{D}$ are true in a given interpretation $\mathcal{A}$, then the conclusion is true in $\mathcal{A}$ as well. But the conclusion is $\top$ which is true in every interpretation, so the rule $\top I$ is sound.

(b) Consider a derivation ending with the $\vee E$ rule:

$$\mathcal{D}: \quad \dfrac{\begin{array}{c}\mathcal{D}'\\ \varphi\vee\psi\end{array} \qquad \begin{array}{c}[\varphi]\\ \mathcal{D}_1\\ \sigma\end{array} \qquad \begin{array}{c}[\psi]\\ \mathcal{D}_2\\ \sigma\end{array}}{\sigma}\ \vee E$$

Let $\mathcal{A}$ be an interpretation in which all undischarged assumptions of $\mathcal{D}$ are true. We want to prove that σ is also true in $\mathcal{A}$.
The rule $\vee E$ does not discharge assumptions in $\mathcal{D}'$, therefore all undischarged assumptions of $\mathcal{D}'$ are also undischarged assumptions of $\mathcal{D}$, hence they are true in $\mathcal{A}$. By induction hypothesis we get that $\varphi\vee\psi$ is true in $\mathcal{A}$, which means that φ is true in $\mathcal{A}$ or ψ is true in $\mathcal{A}$.
If φ is true in $\mathcal{A}$, then all undischarged assumptions of $\mathcal{D}_1$ are true in $\mathcal{A}$, because an undischarged assumption of $\mathcal{D}_1$ is either φ (which is true in $\mathcal{A}$) or an undischarged assumption of $\mathcal{D}$ (which are true in $\mathcal{A}$). Therefore by the induction hypothesis, σ is true in $\mathcal{A}$.
Similarly for $\mathcal{D}_2$, if ψ is true in $\mathcal{A}$, then σ is true in $\mathcal{A}$.
We proved that in all cases, σ is true in $\mathcal{A}$, which concludes the proof that $\vee E$ is sound.

Problem part

8. (a) The first one, $\forall x(\varphi \to \psi) \models (\forall x\varphi) \to (\forall x \psi)$, is true for all formulas φ and ψ. By the soundness theorem it is enough to give a derivation of $\forall x(\varphi \to \psi) \vdash (\forall x \varphi) \to (\forall x \psi)$. Here is one:

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x(\varphi \to \psi)}{\varphi \to \psi}\,\forall E \qquad \dfrac{[\forall x\,\varphi]^1}{\varphi}\,\forall E}{\psi}\,\to E}{\forall x\,\psi}\,\forall I}{(\forall x\varphi) \to (\forall x\,\psi)}\,\to I, 1.$$

The second one is not true for all formulas φ and ψ. For instance take $\varphi = P_1(x)$ and $\psi = P_2(x)$ and consider the interpretation $\mathcal{A}$ such that $|\mathcal{A}| = \{1, 2, 3\}$, $P_1^{\mathcal{A}}$ is true for 1 and false for 2 and 3, and $P_2^{\mathcal{A}}$ is true for 2 and false for 1 and 3.
Then both $\forall x\varphi$ and $\forall x \psi$ are false in $\mathcal{A}$, because $P_1^{\mathcal{A}}$ and $P_2^{\mathcal{A}}$ are not true for all elements of the domain, hence $\forall x\varphi \to \forall x \psi$ is true in $\mathcal{A}$.
On the other hand, $\forall x(\varphi \to \psi)$ is false in $\mathcal{A}$ because for $x=1$, φ is true in $\mathcal{A}$ while ψ is false in $\mathcal{A}$.

Therefore for this example of formulas φ and ψ $(\forall x\varphi) \to (\forall x \psi) \not\models \forall x(\varphi \to \psi)$.

(b) If we take $\varphi = \bot$, then both formulas $(\forall x\varphi) \to (\forall x\psi)$ and $\forall x(\varphi \to \psi)$ are true in all interpretations, therefore $(\forall x\varphi) \to (\forall x\psi) \approx \forall x(\varphi \to \psi)$.

9. (a) The formula χ states that the domain has exactly two elements.

(b) Derivation of $\varphi \vdash \psi$:

$$\dfrac{\dfrac{\dfrac{\forall x_0 \left(f_1(f_1(x_0)) = x_0\right)^{(=\varphi)}}{f_1(f_1(x_0)) = x_0}\;\forall E}{\exists x_1\, f_1(x_1) = x_0}\;\exists I}{\forall x_0 \exists x_1\, f_1(x_1) = x_0 \quad (=\psi)}\;\forall I$$

Derivation of $\varphi \vdash \sigma$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{(=\varphi)}{\forall x_0 (f_1(f_1(x_0)) \doteq x_0)}}{f_1(f_1(x_1)) \doteq x_1}\;\forall E \qquad \dfrac{[f_1(x_0) \doteq f_1(x_1)]^1 \qquad \dfrac{\dfrac{\overset{(=\varphi)}{\forall x_0 (f_1(f_1(x_0)) \doteq x_0)}}{f_1(f_1(x_0)) \doteq x_0}\;\forall E \qquad \overset{\text{refl}}{\dfrac{}{f_1(f_1(x_0)) \doteq f_1(f_1(x_0))}}}{x_0 \doteq f_1(f_1(x_0))}\;\text{repl}}{x_0 \doteq f_1(f_1(x_1))}\;\text{repl}}{x_0 \doteq x_1}\;\text{repl}}{f_1(x_0) \doteq f_1(x_1) \to x_0 \doteq x_1}\;\to I, 1}{\forall x_1 (f_1(x_0) \doteq f_1(x_1) \to x_0 \doteq x_1)}\;\forall I}{\forall x_0, x_1 (f_1(x_0) \doteq f_1(x_1) \to x_0 \doteq x_1) \quad (=\sigma)}\;\forall I$$

Derivation of $\varphi \vdash \psi \wedge \sigma$:

given above

$$\dfrac{\begin{matrix}\vdots \\ \psi\end{matrix} \qquad \begin{matrix}\vdots \\ \sigma\end{matrix}}{\psi \wedge \sigma}\;\wedge I$$

(c) Consider the interpretation $\mathcal{A}$ such that $|\mathcal{A}| = \mathbb{Z}$ and $f_1^{\mathcal{A}}(n) = n+1$.

- ψ is true in $\mathcal{A}$, because for every $m \in \mathbb{Z}$, there exists $n \in \mathbb{Z}$ such that $n+1 = m$, namely $n := m-1$
- σ is true in $\mathcal{A}$, because for every $n, m \in \mathbb{Z}$, if $n+1 = m+1$, then $n = m$.
- φ is not true in $\mathcal{A}$, because it is not the case that $(n+1)+1 = n$ for all $n \in \mathbb{Z}$.

Therefore $\psi, \sigma \nvDash \varphi$ (we just gave a countermodel), hence by the soundness theorem, $\psi, \sigma \vdash \varphi$ is <u>not</u> derivable.

(d) By the completeness theorem it is enough to prove $\psi, \sigma, \chi \vDash \varphi$.
Consider an interpretation $\mathcal{A}$ in which ψ, σ and χ hold, and we have to show that φ holds as well.
We know that $|\mathcal{A}|$ has exactly two elements (because χ is true in $\mathcal{A}$), let's call them a and b, and that $f_1^{\mathcal{A}} : |\mathcal{A}| \to |\mathcal{A}|$ is surjective and injective (because ψ and σ hold).
Given that $f_1^{\mathcal{A}}$ is injective, we have $f_1^{\mathcal{A}}(a) \neq f_1^{\mathcal{A}}(b)$, so there are exactly two possibilities for $f_1^{\mathcal{A}}$:
Either $f_1^{\mathcal{A}}(a) = a$ and $f_1^{\mathcal{A}}(b) = b$
or $f_1^{\mathcal{A}}(a) = b$ and $f_1^{\mathcal{A}}(b) = a$

In both cases we have $f_1^{\mathcal{A}}(f_1^{\mathcal{A}}(a)) = a$ and $f_1^{\mathcal{A}}(f_1^{\mathcal{A}}(b)) = b$, therefore φ is true in $\mathcal{A}$, which is what we wanted to prove. Therefore $\psi, \sigma, \chi \vdash \varphi$ is derivable.

10. (a) $\varphi := \neg \exists x_1 \; P_1(x_1, x_0)$

φ states that there is no natural number strictly smaller than x_0, which is equivalent to $x_0 = 0$.

(b) $\psi := P_1(x_0, x_1) \wedge \neg \exists x_2 \left(P_1(x_0, x_2) \wedge P_1(x_2, x_1) \right)$

ψ states that x_1 is strictly larger than x_0, and that there is no number strictly between x_0 and x_1, which is equivalent to saying that x_1 is the successor of x_0.

(c) φ is always false in $\mathcal{A}'$ because no matter what the value of $v(x_0)$ is, there is always a real number strictly smaller (for instance $v(x_0)-1$).
ψ is always false in $\mathcal{A}'$ as well because if $v(x_0) < v(x_1)$, then there is a real number strictly between them (for instance $\frac{v(x_0)+v(x_1)}{2}$).