MATEMATISKA INSTITUTIONEN
STOCKHOLM UNIVERSITET
Avd. Matematik
Examinator: Yishao Zhou

Tentamensskrivning i
Matematisk modellering, GN
March 22, 2019

A calculator provided by the department is allowed.

There are 5 problems and 6 points each. The grade E requires 15p, D 18p, etc. including maximum 3 bonus points. The problems are not ordered in difficulty level.

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(1) Consider an experiment of mating rabbits. We watch the evolution of a particular gene that appears in two types, G or g. A rabbit has a pair of genes, either GG (dominant), Gg (hybrid, the order is irrelevant, so gG is the same as Gg) or gg (recessive). In mating two rabbits, the offspring inherits a gene from each of its parents with equal probability. Thus, if we mate a dominant (GG) with a hybrid (Gg), the offspring is dominant with probability 1/2 or hybrid with probability 1/2. Start with a rabbit of given character (GG, Gg, or gg) and mate it with a hybrid. The offspring produced is again mated with a hybrid, and the process is repeated through a number of generations, always mating with a hybrid.

(a) Write down the transition probabilities of the Markov chain thus defined.

(b) Assume that we start with a hybrid rabbit. Let μ_n be the probability distribution of the character of the rabbit of the n-th generation. In other words, $\mu_n(GG)$, $\mu_n(Gg)$, $\mu_n(gg)$ are the probabilities that the n-th generation rabbit is GG, Gg, or gg, respectively. Compute μ_1, μ_2, μ_3. Can you do the same for μ_n for general n?

(2) Consider the SIR model

$$\frac{dS}{dt} = -\beta IS$$
$$\frac{dS}{dt} = \beta IS - \alpha I$$
$$\frac{dR}{dt} = \alpha I$$

where $\alpha > 0, \beta > 0$ and S, I and R stand the numbers of susceptible, infected and removed categories of individuals. Assume that $S(0), I(0)$ and $R(0)$ are given and $I(0) \neq 0$.

(a) Show that the the limits of $S(t)$, $R(t)$ and $I(t)$ exist as $t \to \infty$. Determine them.

(b) Determine the ratio β/α in terms of the initial data and/or limits.

(c) Determine $I_{\max}$.

(d) Draw roughly the curve $I(t)$.

(3) Consider the the model for growth under nutrient limitations: $\frac{dN}{dt} = \kappa(C_0 - \alpha N)N$ where we have three parameters C_0, α and κ.

(a) Draw the graph of $N(t)$ without actually solving the equation in terms of stationary points, monotonicity, convexity and initial values.

(b) Determine a dimensionless model with as little parameters as possible. What is your conclusion of dynamical behaviors comparing with those of the original model?

(4) Convert the following linear programming problem

$$\begin{aligned} \min \quad & |x| + |y| + |z| \\ \text{s.t.} \quad & x + y \leq 1, \\ & 2x + z = 3 \end{aligned}$$

to the standrad form $\min c^T x$, subject to $Ax = b(\geq 0), X \geq 0$. Describe how to find a lower bound.

(5) Show that the height of an m-pattern $(a_1, a_2, ..., a_n)$ is at most $m + n - 1$.

You can pick up your paper in room 403, house 6, on March 28 between 12h15 and 12h30.