Solution to the exam Mathematical modeling 2019-03-22

(1) The set of states is $S = \{GG, Gg, gg\}$ with the following transit probabilities:

	GG	Gg	gg
GG	.5	.5	0
Gg	.25	.5	.25
gg	0	.5	.5

So the transition matrix is

$$P = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 1 \end{pmatrix}$$

The elements from the second row of the matrix P^n will give us the probabilities for a hybrid to give dominant, hybrid or recessive species in $(n-1)$th generation in this experiment, respectively (reading this row from left to right). We first find

$$P^2 = \frac{1}{2^2}\begin{pmatrix} 1.5 & 2 & 0 \\ 1 & 2 & 1 \\ 0.5 & 2 & 1.5 \end{pmatrix}, P^3 = \frac{1}{2^3}\begin{pmatrix} 2.5 & 4 & 1.5 \\ 2 & 4 & 2 \\ 1.5 & 4 & 2.5 \end{pmatrix}, P^4 = \frac{1}{2^4}\begin{pmatrix} 4.5 & 8 & 3.5 \\ 4 & 8 & 4 \\ 3.5 & 8 & 4.5 \end{pmatrix}.$$

We can show, by induction or diagonalizing the matrix P) that

$$P^n = \frac{1}{2^n}\begin{pmatrix} \frac{3}{2} + (2^{n-2} - 1) & 2^{n-1} & \frac{1}{2} + (2^{n-2} - 1) \\ 2^{n-2} & 2^{n-1} & 2^{n-2} \\ \frac{1}{2} + (2^{n-2} - 1) & 2^{n-1} & \frac{3}{2} + (2^{n-2} - 1) \end{pmatrix}, n = 1, , 2, 3, ...$$

So that

$$\mu_i(GG) = 0.25, \mu_i(Gg) = 0.5, \mu_i(gg) = 0.25, i = 1, 2, 3.$$

(In fact the left eigenvector of P.)

(2) Note that $\frac{dS}{dt} < 0$ for all t, the number of susceptible individuals is always declining, independently of the initial condition $S(0) > 0$. Since $S(t)$ is monotone and positive and bounded below, we have $S(t) \to S_\infty$ as $t \to \infty$. Next, The number of recovered individuals also has monotone behavior, independently of the initial conditions. Since $\frac{dR}{dt} > 0$ for all t, the number of recovered individuals is always increasing. Since the number of recovered is monotone and bounded above by $N = S(t) + I(t) + R(t)$ for all t, we have $R(t) \to R_\infty$ as $t \to \infty$.

On the other hand, the number of infected individuals may be monotonically decreasing to zero, or may have monotone behavior by first increasing to some maximum level, and then decreasing to zero. The prevalence first starts increasing if $\frac{dI}{dt}(0) = (\beta S(0) - \alpha)I(0) > 0$. Hence, a necessary and sufficient condition for an initial increase in the number of infected is $\beta S(0) - \alpha > 0$, or $\beta S(0)\alpha > 1$.

To determine the limits S_∞ and R_∞, we simply solve S in terms of R by dividing the equation for S and the equation for R. Hence

$$\frac{dS}{dR} = -\frac{\beta}{\alpha}S.$$

Solving, we get

$$S = S(0)e^{-\frac{\beta}{\alpha}R} \geq S(0)e^{-\frac{\beta}{\alpha}N} > 0$$

proving that $S_\infty > 0$. Now we show that the epidemic dies out. If $I(t) \to I_\infty$ then $I_infty = 0$. To see this we integrate the S-equation in the model:

$$\int_0^\infty \frac{dS}{dt} dt = -\beta \int_0^\infty S(t)I(t)dt$$
$$S_\infty - S_0 = -\beta \int_0^\infty S(t)I(t)dt$$
$$S_0 - S_\infty \geq \beta S_\infty \int_0^\infty I(t)dt$$

The last equation shows that $I(t)$ is integrable on $[0, \infty)$. Hence the limit of $I(t)$ exists as $t \to \infty$. And it is 0.

Now we solve I, S by dividing the equation for I and the equation for S:

$$\frac{sI}{dS} = \frac{\beta SI - \alpha I}{-\beta SI} = -1 + \frac{\alpha}{\beta S}$$

Separating S and I and integrating yield the implicit solution

$$I = -S + \frac{\alpha}{\beta} \ln S + C \quad \Leftrightarrow \quad I + S - \frac{\alpha}{\beta} \ln S = C \text{ for } allt$$

where C is an integral constant. Since $S_\infty > 0$ and $I_\infty = 0$ we have

$$I(0) + S(0) - \frac{\alpha}{\beta} \ln S(0) = S_\infty - \frac{\alpha}{\beta} - \ln S_\infty.$$

Therefore

$$\frac{\beta}{\alpha} = \frac{\ln(S(0)/S_\infty)}{S(0) + I(0) - S_\infty}$$

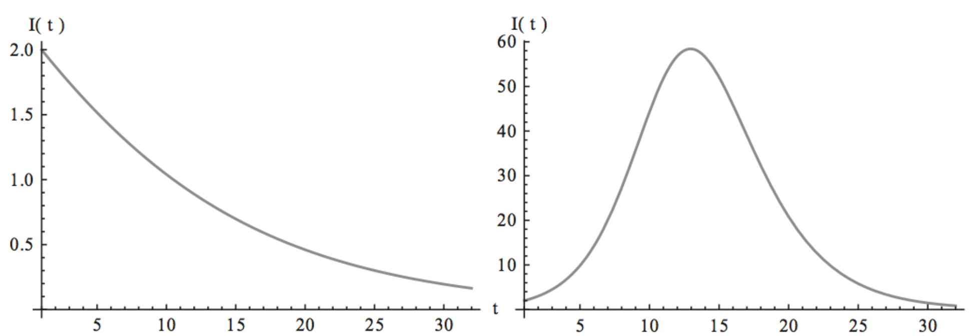
Note that since $S(t)$ is a decreasing function, we have $S_\infty < S(0) + I(0)$. The implicit solution also allows us to compute the maximum number of infected individuals that is attained. This number occurs when $\frac{dI}{dt} I = 0$, that is, when $S = \frac{\alpha}{\beta}$. From

$$I + S - \frac{\alpha}{\beta} \ln S = I(0) + S(0) - \frac{\alpha}{\beta} \ln S(0)$$

substituting the expression for S and moving all terms but I to the right-hand side leads to

$$I_{\max} = -\frac{\alpha}{\beta} + \frac{\alpha}{\beta} \ln \frac{\alpha}{\beta} + S(0) + I(0) - \frac{\alpha}{\beta} \ln S(0)$$

Below two typical situation for $I(t)$ are depicted. Left: shows the prevalence monotonically decreasing. Right: shows the prevalence first increasing and then decreasing to zero.

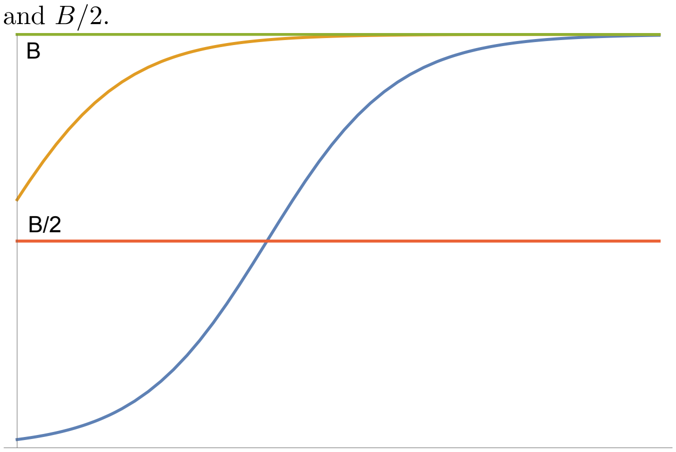


(3) First we note that there are two steady states (or equilibria) at $N = 0$ and $N = B := B$. We call this number It is clear that N is either decreasing or increasing depending on the sign of dN/dt, that is the sign of $C_0 - \alpha N$ since $N(t) \geq 0$. Therefore $N(t)$ increases if $N < B$ and $N(t)$ decreases if $N > B$. So if we start with $N(0) > B$ then $N(t)$ will monotonically decreases to B as $t \to \infty$ (never crosses $N = B$). Similarly, if we start with $N(0) < B$ then $N(t)$ will monotonically increases to B as $t \to \infty$ (never crosses $N = B$).

Now we study the sign of the second derivative of N to decide the convexity.

$$\frac{d^2N}{dt^2} = \kappa(C_0 - 2\alpha N)\frac{dN}{dt} = \kappa(C_0 - 2\alpha N)(C_0 - \alpha N)N.$$

Obviously, $N(t)$ is concave if N is between $B/2$ and B, and convex when N is between 0 and $B/2$.



To reduce the number of parameters we do the variable scaling. Let $N = \hat{N}N^*$ and $t = \hat{t}t^*$, where stars indicate new variables and the hats are constants to be chosen. Proceeding purely formally, we substitute these into the differential equation:

$$\frac{d(\hat{N}N^*)}{d(\hat{t}t^*)} = \kappa(C_0 - \alpha\hat{N}N^*)\hat{N}N^*$$
$$\Rightarrow \frac{\hat{N}}{\hat{t}}\frac{dN^*}{dt^*} = \kappa(C_0 - \alpha\hat{N}N^*)\hat{N}N^*$$
$$\Rightarrow \frac{dN^*}{dt^*} = (\kappa\hat{t}C_0 - \kappa\hat{t}\alpha\hat{N}N^*)N^*$$

Let us look at this last equation: we?d like to make $\kappa\hat{t}C_0 = 1$ and $\kappa\hat{t}\alpha\hat{N} = 1$. This gives us $\hat{t} = 1/(\kappa C_0)$ and $\hat{N} = 1/(\kappa\hat{t}\alpha)$. So the equation is reduced to

$$\frac{dN^*}{dt^*} = (1 - N^*)N^*$$

We see that the dynamical behavior will be the same as that of the original model.

(4) To make the objective function linear $u + v + w$ we need $u, v, w \geq 0$ and $|x| \leq u$, $|y| \leq v$, and $|z| \leq w$; to make all variables nonnegative we introduce $x^{\pm} \geq 0$, $y^{\pm} \geq 0$ and $z^{\pm} \geq 0$ so $x = x^+ - x^-$, $y = y^+ - y^-$, and $z = z^+ - z^-$. So we get the constraints:

$$x^+ - x^- - u \leq 0, \quad -(x^+ - x^-) - u \leq 0,$$
$$y^+ - y^- - v \leq 0, \quad -(y^+ - y^-) - v \leq 0,$$
$$z^+ - z^- - w \leq 0, \quad -(z^+ - z^-) - w \leq 0,$$
$$x^+ - x^- + y^+ - y^- \leq 1, \quad 2x^+ - 2x^- + z^+ - z^- = 3$$

To make all inequalities to equality we introduce the slack variables $s_1, .., s_7 \geq$ so that

$$\begin{gathered}
x^+ - x^- - u + s_1 = 0, \quad -(x^+ - x^-) - u + s_2 = 0, \\
y^+ - y^- - v + s_3 = 0, \quad -(y^+ - y^-) - v + s_4 = 0, \\
z^+ - z^- - w + s_5 = 0, \quad -(z^+ - z^-) - w + s_6 = 0, \\
x^+ - x^- + y^+ - y^- + s_7 = 1, \quad 2x^+ - 2x^- + z^+ - z^- = 3
\end{gathered}$$

Now let $X^T = (u, v, w, x^+, x^- y^+, y^-, z^+, z^-, s_1, ..., s_7) \in \mathbb{R}^{16}_+$, $b^T = (0, 0, 0, 0, 0, 0, 1, 3)$, and $c^T = (1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$

$$A = \begin{pmatrix}
-1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 2 & -2 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \in \mathbb{R}^{8 \times 16}$$

The lower bound can be obtained by the dual problem which is to maximize $b^T \lambda$ subject to $A^T \lambda \leq c$ where $\lambda \in \mathbb{R}^8$, (note that there is no sign constraints).

(5) It is enough to show that the height cannot be greater than $m + n - 1$. Suppose, in contradiction, that the height is greater than $m + n - 1$. Then there is at least one entry, say a_i, that is greater than $m + n - 1$. Since we have $i \geq 1$ and $a_i > m + n - 1$, we get $a_i + i > m + n - 1 + 1 = m + n$. But this contradicts the fact that $(a_1, a_2, ..., a_n)$ is an $m-$pattern.