Solution to the exam Mathematical modeling 2019-03-22

(1) The Markov chain in this exercise has the following set states

$$S = \{\text{Professional, Skilled, Unskilled}\}$$

with the following transition probabilities:

	Professional	Skilled	Unskilled
Professional	0.8	0.1	0.1
Skilled	0.2	0.6	0.2
Unskilled	0.25	0.25	0.5

so that the transition matrix for this chain is

$$P = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.25 & 0.25 & 0.5 \end{pmatrix}$$

Then

$$P^2 = \begin{pmatrix} 0.685 & 0.165 & 0.15 \\ 0.33 & 0.43 & 0.24 \\ 0.2375 & 0.3 & 0.325 \end{pmatrix}$$

and thus the probability that a randomly chosen grandson of an unskilled labourer is a professional man is 0.375.

(2) Since $(a-1, ..., a_n)$ is an m-pattern, we have

$$\{a_1+1, a_2+2, ..., a_n+n\} = \{m+1, m+2, ..., m+n\}.$$

It follows that

$$(a_1+1)+(a_2+2)+\cdots+(a_n+n) = (m+1)+(m+2)+\cdots+(m+n)$$
$$\Leftrightarrow a_1+a_2+\cdots a_n = mn.$$

(3) Add the two equation we have $d(I+S)/dt = 0$ so $S(t)+I(t)$ for all t. equals a constant which we assume is N and it is the same as $S(0)+I(0)$. So $S = N - I$. Substituting it in the second equation yields

$$\frac{dI}{dt} = \beta I(N-I) - \alpha I$$

This can be written as

$$I'(t) := \frac{dI}{dt} = rI\left(1 - \frac{I}{K}\right)$$

where $r = \beta N - \alpha$ and $K = r/\beta$. The parameter r is often referred to as the growth rate. We can see that r can be positive or negative, so we consider two cases.

(i) $r < 0$: If the growth rate is negative, $r < 0$, then the number of infected individuals $I(t)$ tends to 0 as $t \to \infty$. To see this, notice that if $r < 0$, then $K < 0$. Hence, $I'(t) \le rI(t)$. The solutions of this simple differential inequality are $I(t) \le I(0)e^{rt}$, and they approach zero for $r < 0$. This implies that if $r < 0$, the disease gradually disappears from the population on its own.

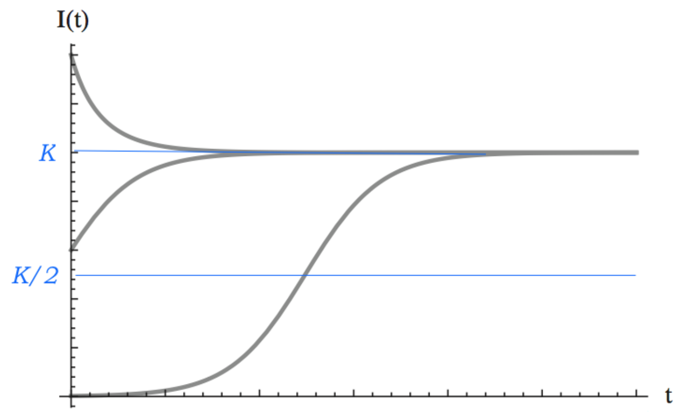
(ii) $r > 0$: The logistic equation can be solved, and in this case, we can solve it explicitly. However it is not necessary. First we note that there are two steady states at $I = 0$ and $I = K$. It is clear that I is either decreasing or increasing depending on the sign of $I'(t)$, that is the sign of $1 - I/K$ since $I(t) \ge 0$. Therefore $I(t)$ increases if $I < K$ and $I(t)$

decreases if $I > K$. So if we start with $I(0) > K$ then $I(t)$ will monotonically decreases to K as $t \to \infty$ (never crosses $I = K$). Similarly, if we start with $I(0) < K$ then $I(t)$ will monotonically increases to K as $t \to \infty$ (never crosses $I = K$).

Now we study the sign of the second derivative of I to decide the convexity.

$$\frac{d^2 I}{dt^2} = r\left(1 - \frac{2I}{K}\right)\frac{dI}{dt} = r^2\left(1 - \frac{2I}{K}\right)\left(1 - \frac{I}{K}\right)$$

For solutions in the interval $0 < I(t) < K$, the second derivative changes sign when I crosses $I = K/2$). Thus values of t such that $I(t) < K/2$, the second derivative of I is positive, and $I(t)$ is convex. For values of t for which $I(t) > K/2$, the second derivative of I is negative so $I(t)$ is concave.



(4) We start by writing $N = \hat{N}N^*$, $C = \hat{C}C^*$ and $t = \hat{t}t^*$, and substitute in the equations we can define $\hat{C} = k_n$, $\hat{t} = \frac{V}{F}$ and $\hat{N} = \frac{k_n F}{\alpha V k}$. Introduce $\alpha_1 = \frac{Vk}{F}$, and $\alpha_2 = \frac{C_0}{k_n}$ we end up with

$$\frac{dN^*}{dt^*} = \alpha_1 \frac{C^*}{1 + C^*}N^* - N^*$$
$$\frac{dC^*}{dt^*} = -\frac{C^*}{1 + C^*}N^* - C^* + \alpha_2$$

Now we drop $*$ in the equations:

$$\frac{dN}{dt} = \alpha_1 \frac{C}{1 + C}N - N$$
$$\frac{dC}{dt} = -\frac{C}{1 + C}N - C + \alpha_2$$

Define $f(N, C) := \alpha_1 \frac{C}{1+C}N - N$, $g(N, C) := -\frac{C}{1+C}N - C + \alpha_2$.

Solving $f(N, C) = 0, g(N, C) = 0$ yields two equilibria (steady states):

$$\bar{X}_1 = (0, \alpha_2), \quad \bar{X}_2 = \left(\alpha_1\left(\alpha_2 - \frac{1}{\alpha_1 - 1}\right), \frac{1}{\alpha_1 - 1}\right)$$

An equilibrium is physically meaningful only if $\bar{\geq} 0$ and $\bar{N} \geq 0$. So $\bar{X}_1$ is always well-defined in this sense but not the second. The equilibrium $\bar{X}_2$ is well-defined and makes physical sense only if

$$\alpha_1 > 1 \text{ and } \alpha_2 > \frac{1}{\alpha_1 - 1} \quad \Longleftrightarrow \alpha_1 > 1 \text{ and } \alpha_2(\alpha_1 - 1) > 1$$

At any point (N, C) the Jacobian $A = F'$ of $F = (f(N, C), g(N, C))^\top$ is

$$A = \begin{pmatrix} \alpha_1 \frac{C}{1+C} - 1 & \frac{\alpha_1 N}{(1+C)^2} \\ -\frac{C}{1+C} & -\frac{N}{(1+C)^2} - 1 \end{pmatrix}$$

In particular, at $\bar{X}_2$ where $\bar{C} = \frac{1}{\alpha_1 - 1}$, $\bar{N} = \frac{\alpha_1(\alpha_1\alpha_2 - \alpha_2 - 1)}{\alpha_1 - 1}$ we have

$$A_2 := F'(\bar{X}_2) = \begin{pmatrix} 0 & \beta(\alpha_1 - 1) \\ -\frac{1}{\alpha_1} & -\frac{\beta(\alpha_1 - 1) + \alpha_1}{\alpha_1} \end{pmatrix}$$

where $\beta = \alpha_2(\alpha_1 - 1) - 1 > 0$. The trace of A_2 is $-\frac{\beta(\alpha_1 - 1) + \alpha_1}{\alpha_1} < 0$, and the determinant of A_2 is $\frac{\beta(\alpha_1 - 1)}{\alpha_1} > 0$, if $\alpha_1 > 1$. So $\bar{X}_2$ is a locally stable positive equilibrium.

Now at $\bar{X}_1$ $\bar{N} = 0$ and $\bar{C} = \alpha_2$.

$$A_1 := F'(\bar{X}_1) = \begin{pmatrix} \frac{\beta}{1+\alpha_2} & 0 \\ -\frac{\alpha_2}{1+\alpha_2} & -1 \end{pmatrix}$$

and thus we see that its determinant is

$$-\frac{\beta}{1 + \alpha_2} < 0$$

and therefore the steady state $\bar{X}_1$ is unstable. It turns out that this is a saddle point.

(5) (a) This is the least squares: We have $Ac = b$ with $A = 1, 2, 3)^T$ and $b = (2, 5, 8)^T$. Using normal equation $A^T A c = A^T b$ which is $14c = 36$ we have $c = 18/7$.

(b) This can be converted to an LP problem: min r s.t. $r \geq 0$ and

$$r - (2 - c) \geq 0, r + (2 - c) \geq 0,$$
$$r - (5 - 2c) \geq 0, r + (5 - 2c) \geq 0,$$
$$r - (8 - 3c) \geq 0, r + (8 - 3c) \geq 0$$

Solve is graphically we get $c = 5/2$ and $r = 1/2$.