MATEMATISKA INSTITUTIONEN
STOCKHOLM UNIVERSITET
Avd. Matematik
Examinator: Yishao Zhou

Tentamensskrivning i
Matematisk modellering, GN
March 22, 2019

A calculator provided by the department is allowed.

There are 5 problems and 6 points each. The grade E requires 15p, D 18p, etc. including maximum 3 bonus points. The problems are not ordered in difficulty level.

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(1) Assume that a man's profession can be classified as professional, skilled labourer, or unskilled labourer. Assume that, of the sons of professional men, 80 percent are professional, 10 percent are skilled labourers, and 10 percent are unskilled labourers. In the case of sons of skilled labourers, 60 percent are skilled labourers, 20 percent are professional, and 20 percent are unskilled. Finally, in the case of unskilled labourers, 50 percent of the sons are unskilled labourers, and 25 percent each are in the other two categories. Assume that every man has at least one son, and form a Markov chain by following the profession of a randomly chosen son of a given family through several generations. Set up the transition matrix. Find the probability that a randomly chosen grandson of an unskilled labourer is a professional man.

(2) Let $(a_1, a_2, ..., a_n)$ be an m-pattern. What is $a_1 + \cdots + a_n$?

(3) Consider a simple SIS epidemic model

$$\frac{dS}{dt} = -\beta IS + \alpha I$$
$$\frac{dI}{dt} = \beta IS - \alpha I$$

where $\alpha > 0, \beta > 0$ and S, and I stand the numbers of susceptible and infected categories of individuals. Assume that $S(0)$ and $I(0)$ are given and $I(0) \neq 0$.

(a) Show that $I(t) + S(t)$ is a constant for all $t \geq 0$.
(b) Show that this model can be reduced to the logistic equation in the form $\frac{dI}{dt} = rI(1 - I/K)$. Note that r and K are new parameters which can be decided from your derivation.
(c) Explain the meaning of $r < 0$ and $r > 0$.
(d) Draw the graph of $I(t)$ without actually solving the equation in terms of stationary points, monotonicity, convexity and initial values.

(4) A mathematical model for the chemostat is given as follows:

$$\frac{dN}{dt} = \frac{kC}{k_n + C}N - \frac{F}{V}N$$
$$\frac{dC}{dt} = -\alpha\frac{kC}{k_n + C}N - \frac{F}{V}C + \frac{F}{V}C_0$$

where we have 6 parameters k, k_n, F, V, α and C_0.

(a) Reduce the the number of parameters as many as you can.
(b) Determine the steady state(s) and the conditions for it/them being physically meaningful?
(c) Describe the local stability of the steady state(s).

(5) Fit the model $y = cx$ to the following data set:

x	1	2	3
y	2	5	8

Formulate mathematical programs to find "best" values of c by minimizing each of the following criteria:

a) $\sum_{i=1}^{3}(y_i - y(x_i))^2$; b) $\max_{1\leq i\leq 3} |y_i - y(x_i)|$

Which formulations can be converted to linear programming problems and do it.

You can pick up your paper in room 403, house 6, on May 2 between 12h15 and 12h30.