MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Tentamensskrivning i
Ordinary Differential Equations
7.5 hp
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No calculators, books, or other resources allowed. Max score is 30p; grade of E guaranteed at 15p. Appropriate amount of details required for full marks.

1. **(6p)** Find all solutions to the differential equation $xy' + (x-2)y = x^4$.

   Solution. Thanks to the superposition principle it suffices to find solutions to the associated homogeneous problem and one particular solution. Assuming $x \neq 0$ we can put the homogeneous problem in the form

   $$y' + (1 - \frac{2}{x})y = 0,$$

   which is solved by the functions,

   $$y_h(x) = e^{\int -1 + \frac{2}{x}\,dx} = e^{-x + 2\ln x + c} = Ce^{-x}x^2.$$

   Note that the last formula makes sense for $C, x \in \mathbb{R}$. We check that it indeed solves the homogeneous problem $xy' + (x-2)y = 0$. By linearity it suffices to do so for $C = 1$, which is verified through the following calculations.

   $$\begin{aligned} xy_h'(x) &= x(2xe^{-x} - x^2e^{-x}) = (2x^2 - x^3)e^{-x} \\ (x-2)y_h(x) &= (x^3 - 2x^2)e^{-x}. \end{aligned}$$

   Next, we are going to find a particular solution. The form of the differential equation suggests a polynomial ansatz. If y is a polynomial function of order n, then the left hand side $xy' + (x-2)y$ is a polynomial of order $n+1$. So we try the ansatz

   $$y(x) = x^3 + \alpha x^2 + \beta x + \gamma$$

   leading to the equations

   $$\begin{aligned} \alpha + 1 &= 0 \\ \beta &= 0 \\ \gamma - \beta &= 0. \end{aligned}$$

   So one particular solution to the differential equation is

   $$y_p(x) = x^3 - x^2.$$

   It follows that the general solution has the form

   $$y(x) = y_h(x) + y_p(x) = Ce^{-x}x^2 + x^3 - x^2$$

   for $C, x \in \mathbb{R}$.

2. **(4p)** Determine the general, real solution to the system

   $$\begin{Bmatrix} x' = & 2x - y \\ y' = & x \end{Bmatrix}$$

Solution. It is key to observe that the given system of differential equations can be reformulated as the second order differential equation

$$y'' = 2y' - y$$

or equivalently

$$y'' - 2y' + y = 0.$$

The last differential equation can be solved by considering its characteristic equation. We have

$$X^2 - 2X + 1 = (X-1)^2$$

so that a fundamental system of solutions is given by

$$\begin{aligned} y_1(t) &= e^t \\ y_2(t) &= te^t. \end{aligned}$$

Using the relation $y' = x$, we then find

$$\begin{aligned} x_1(t) &= y_1'(t) = e^t \\ x_2(t) &= y_2'(t) = e^t + te^t. \end{aligned}$$

So the general solution of the given system of differential equations is parametrised by $c_1, c_2 \in \mathbb{R}$

$$\begin{pmatrix} x \\ y \end{pmatrix}(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1+t \\ 1 \end{pmatrix}$$

3. **(7p)** Use the power series methods to find the solution to the initial value problem

$$\begin{Bmatrix} x^2y'' + xy' + x^2y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{Bmatrix}$$

Solution. The power series method uses the ansatz

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

for real coefficients $a_k \in \mathbb{R}$ indexed by $k \in \mathbb{N}$. Since formal differential applies we get

$$\begin{aligned} y''(x) &= \sum_{k=0}^{\infty} (k+1)(k+2)a_{k+2}x^k \\ y'(x) &= \sum_{k=0}^{\infty} (k+1)a_{k+1}x^k. \end{aligned}$$

Calculating the relevant terms for the differential equation leads to

$$\begin{aligned} x^2y''(x) &= \sum_{k=2}^{\infty} (k-1)k a_k x^k \\ xy'(x) &= \sum_{k=1}^{\infty} k a_k x^k \\ x^2y(x) &= \sum_{k=2}^{\infty} a_{k-2} x^k \end{aligned}$$

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Comparing coefficients of power series, the differential equation $x^2y'' + xy' + x^2y = 0$ leads to the condition $a_1 = 0$ as well as the recurrence relation

$$\underbrace{(k-1)ka_k + ka_k}_{=k^2 a_k} + a_{k-2} = 0,$$

which holds for all $k \geq 2$. Equivalently,

$$a_{k+2} = -\frac{1}{(k+2)^2} a_k,$$

for all $k \in \mathbb{N}$. This recurrence relation strongly resembles the ones obtained for the differential equation $y' = y$, whose solution is the exponential function. We thus know how to proceed, showing by induction that

$$\begin{aligned} a_{2k} &= a_0(-1)^k \left(\frac{1}{2^k k!}\right)^2 \\ a_{2k+1} &= a_1(-1)^k \left(\frac{2^k k!}{(2k+1)!}\right)^2. \end{aligned}$$

Finally, the initial conditions translate to

$$\begin{aligned} a_0 &= y(0) = 1 \\ a_1 &= y'(0) = 0. \end{aligned}$$

Note that this is compatible with the condition $a_1 = 0$ found earlier on. So we obtain the solution

$$y(x) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{2^k k!}\right)^2 x^{2k}.$$

4. **(7p)** (a) Determine all critical points of the autonomous system

$$\begin{Bmatrix} x' = -e^x y \\ y' = y^2 - x^2 - 2y + 2x \end{Bmatrix}$$

(b) Investigate whether these critical points are asymptotically stable, stable or unstable.

Solution. We first find the critical points, answering (a). Writing

$$\begin{aligned} F(x,y) &= -e^x y \\ G(x,y) &= y^2 - x^2 - 2y + 2x \end{aligned}$$

we have to find those pairs $(x, y) \in \mathbb{R}^2$ satisfing $F(x, y) = 0$ and $G(x, y) = 0$. Starting with the latter equation, we write

$$y^2 - x^2 - 2y + 2x = (y^2 - 2y + 1) - (x^2 - 2x + 1) = (y-1)^2 - (x-1)^2,$$

which equals zero if and only if $|y - 1| = |x - 1|$. Distinguishing the signs of both sides, we end up with two families of solutions

$$\{(x, y) \in \mathbb{R} \mid x = y\} \quad \text{and} \quad \{(x, y) \in \mathbb{R} \mid x + y = 2\}.$$

Substituting $y = x$ and $y = 2 - x$, respectively, into $F(x, y) = 0$ we find the critical points

$$(0, 0) \quad \text{and} \quad (2, 0).$$

This answers (a).

We next solve (b) by linear approximation. To this end, we calculate the Jacobian

$$D_{(x,y)} = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix} = \begin{pmatrix} -e^x y & -e^x \\ -2x+2 & 2y-2 \end{pmatrix}.$$

Substituting $(x, y) = (0, 0)$, we find the matrix

$$D_{(0,0)} = \begin{pmatrix} 0 & -1 \\ 2 & -2 \end{pmatrix},$$

whose characteristic polynomial is

$$\det \begin{pmatrix} -X & 1 \\ 2 & -2-X \end{pmatrix} = (-X)(-2-X) - (-1)2 = X^2 + 2X + 2 = (X^2 + 2X + 1) + 1 = (X+1)^2 + 1.$$

So the roots of this characteristic polynomial are exactly solutions to the equation

$$(X+1)^2 = -1,$$

which are $-1+i$ and $-1-i$. Since both roots have a negative real part, it follows that the critical point $(0, 0)$ is asymptotically stable.

We next consider the critical point $(2, 0)$. The Jacobian at this point is

$$D_{(2,0)} = \begin{pmatrix} 0 & -e^2 \\ -2 & -2 \end{pmatrix}.$$

The product of its eigenvalues equals its determinant

$$\det \begin{pmatrix} 0 & -e^2 \\ -2 & -2 \end{pmatrix} = 0(-2) - (-e^2)(-2) < 0,$$

while their sum is the trace

$$\mathrm{Tr} \begin{pmatrix} 0 & -e^2 \\ -2 & -2 \end{pmatrix} = 0 + (-2) \neq 0.$$

So the two eigenvalues of $D_{(2,0)}$ have non-zero real part, and one of them must have negative real part. So the critical point $(2, 0)$ is unstable.

5. **(6p)** Consider the boundary value problem

$$u'' + 2u' + u = 0 \quad \text{in } [0,1], \qquad u(0) + u'(0) = 0, \quad u(1) - u'(1) = 0. \qquad (*)$$

(a) Show that this boundary value problem has a unique solution.

(b) Find a Sturm-Liouville boundary value problem with the same solutions as $(*)$.

Solution. Before addressing the questions, (a), (b) and (c), we determine a fundamental system for the differential equation $u'' + 2u + u = 0$. The characteristic equation $0 = X^2 + 2X + 1 = (X+1)^2$ shows that

$$u_1(t) = e^{-t}$$
$$u_2(t) = te^{-t}$$

form a fundamental system. Their derivatives are

$$u_1'(t) = -e^{-t}$$
$$u_2'(t) = e^{-t} - te^{-t}.$$

Further the evaluations at 0 and 1, respectively are

$$\begin{array}{ll} u_1(0) = 1 & u_2(0) = 0 \\ u_1'(0) = -1 & u_2'(0) = 1 \\ u_1(1) = \frac{1}{e} & u_2(1) = \frac{1}{e} \\ u_1'(1) = -\frac{1}{e} & u_2'(1) = 0. \end{array}$$

We now turn to solving (a). If the solution $au_1 + bu_2$ of $u'' + 2u + u = 0$ satisfies $u(0) + u'(0) = 0$, then

$$0 = au_1(0) + bu_2(0) + au_1'(0) + bu_2'(0) = a \cdot 1 + b \cdot 0 + a \cdot (-1) + b \cdot 1 = b.$$

Further, the function u_1 satisfies

$$u_1(1) - u_1'(1) = \frac{1}{e} + \frac{1}{e} \neq 0.$$

So indeed the boundary value problem $(*)$ has the unique solution $u \equiv 0$.

In order to find a Sturm-Liouville boundary value problem with the same solutions as $(*)$, we recall that a general equation

$$u'' + a_1(x)u' + a_0(x)$$

has to be multiplied by

$$e^{\int a_1(x)}.$$

In our the concrete case, we find $a_1 \equiv 2$ and thus multiply the differential equation with e^{2x}, leading to the Sturm-Liouville problem

$$(e^{2x}u')' + e^{2x}u = 0 \quad \text{in } [0,1], \qquad u(0) + u'(0) = 0, \quad u(1) - u'(1) = 0.$$