MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Sven Raum

Tentamensskrivning i
Ordinary Differential Equations
7.5 hp
August 21, 2019

No calculators, books, or other resources allowed. Max score is 30p; grade of E guaranteed at 15p. Appropriate amount of details required for full marks.

1. **(10p)** Consider the system of linear differential equations

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} e^x \\ 0 \end{pmatrix} \tag{$*$}$$

and solve the following questions.

(a) Find all solutions to the differential equation $y' = y + e^t$.
(b) Find a fundamental matrix of the homogeneous system associated with $(*)$.
(c) Calculate the Wronskian associated with the fundamental matrix you found.
(d) Apply the Duhamel formula to find a particular solution of $(*)$.

Solution.

(a) For linear first order differential equations, there are explicit formulae for a solution. We have $y' = a(t)y + b(t)$ with $a(t) = 1$ and $b(t) = e^t$. The general solution of this equation is given by

$$y(t) = c\exp\left(\int_0^t a(s)ds\right) + \left(\int_0^t b(\tau)\exp\left(-\int_0^\tau a(s)ds\right)d\tau\right)\left(\exp\left(\int_0^t a(s)ds\right)\right).$$

We calculate the different terms appearing in the this equation:

$$\exp\left(\int_0^t a(s)ds\right) = \exp\left(\int_0^t 1ds\right) = \exp(t)$$

and

$$\int_0^t b(\tau)\exp\left(-\int_0^\tau a(s)ds\right)d\tau = \int_0^t e^\tau e^{-\tau}d\tau = t.$$

It follows that the general solution of $y' = y + e^t$ is

$$y(t) = ce^t + te^t$$

for a real number c.

(b) The homogeneous system of equations associated with $(*)$ can be written as

$$\begin{aligned} y_1' &= y_1 + y_2 \\ y_2' &= y_2. \end{aligned}$$

The second equation has the general solution $y_2(x) = De^x$ for a real number D. Plugging this into the first equation, we obtain $y_1' = y_1 + De^x$, whose general solution can be obtained as in the last item as $y_1(x) = Ce^x + Dxe^x$ for real numbers C and D. So a fundamental matrix of $(*)$ is given by

$$\begin{pmatrix} e^x & xe^x \\ 0 & e^x \end{pmatrix}.$$

(c) The Wronskian is the determinant of the fundamental matrix:

$$\det \begin{pmatrix} e^x & xe^x \\ 0 & e^x \end{pmatrix} = e^x e^x - xe^x 0 = e^{2x}. \tag{1}$$

(d) The Duhamel formula describes a particular solution of $(*)$ as

$$y_p(x) = \phi(x) \int_0^x \phi^{-1}(s) f(x) \tag{2}$$

where ϕ is the fundamental matrix, ϕ^{-1} is the inverse of the fundamental matrix and $f(x)$ is the inhomogeneous term of $(*)$. Concretely, these functions are

$$\phi(x) = \begin{pmatrix} e^x & xe^x \\ 0 & e^x \end{pmatrix}$$

$$\phi^{-1}(x) = \begin{pmatrix} e^x & xe^x \\ 0 & e^x \end{pmatrix}^{-1} = \begin{pmatrix} e^{-x} & -xe^{-x} \\ 0 & e^{-x} \end{pmatrix}$$

$$f(x) = \begin{pmatrix} e^x \\ 0 \end{pmatrix}.$$

Plugging these terms into the Duhamel formula, we obtain

$$\begin{aligned} y_p(x) &= \begin{pmatrix} e^x & xe^x \\ 0 & e^x \end{pmatrix} \int_0^x \begin{pmatrix} e^{-s} & -xe^{-s} \\ 0 & e^{-s} \end{pmatrix} \begin{pmatrix} e^s \\ 0 \end{pmatrix} ds \\ &= \begin{pmatrix} e^x & xe^x \\ 0 & e^x \end{pmatrix} \int_0^x \begin{pmatrix} 1 \\ 0 \end{pmatrix} ds \\ &= \begin{pmatrix} e^x & xe^x \\ 0 & e^x \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} xe^x \\ 0 \end{pmatrix}. \end{aligned}$$

2. **(8p)** Use the Laplace transform methods to find the solution to the initial value problem

$$\begin{Bmatrix} y'' + y = x \\ y(0) = 0 \\ y'(0) = 0 \end{Bmatrix}$$

Solution. We apply the Laplace transform to both sides of the given differential equation, simplify and obtain an expression for the Laplace transform of the solution of the initial value problem:

$$L[y'' + y](p) = L[x](p). \tag{3}$$

The right-hand side of this equation is

$$L[x](p) = \frac{1}{p^2}$$

and its left-hand side can be simplified using the initial conditions

$$L[y'' + y](p) = (p^2 L[y](p) - py'(0) - y(0)) + L[y] = p^2 L[y](p) + L[y](p) = (p^2 + 1)L[y](p).$$

Plugging these calculations into the equation (3), we obtain

$$(p^2 + 1)L[y](p) = \frac{1}{p^2}.$$

Solving this equation for $L[y](p)$ and writing it in terms of known Laplace transforms gives

$$\begin{aligned} L[y](p) &= \frac{1}{p^2(p^2+1)} \\ &= \frac{p^2+1}{p^2(p^2+1)} + \frac{-p^2}{p^2(p^2+1)} \\ &= \frac{1}{p^2} - \frac{1}{p^2+1} \\ &= L[x](p) - L[\sin x](p). \end{aligned}$$

So we find the candidate solution $y(x) = x - \sin(x)$. Indeed, we have

$$y''(x) + y(x) = \sin(x) + x - \sin(x) = x.$$

3. **(6p)**

(a) Consider a general autonomous system of differential equations

$$\begin{Bmatrix} x' = F(x, y) \\ y' = G(x, y) \end{Bmatrix} \tag{$**$}$$

and define the following terms:

- a "critical point" of the system,
- a "stable" critical point of the system, and
- an "asymptotically stable" critical point of the system.

(b) Consider the autonomous system

$$\begin{Bmatrix} x' = \sin(x) + \sin(y) \\ y' = (x+1)(y-1) + 1 \end{Bmatrix} \tag{$***$}$$

Find all its critical points and determine for each of them whether it is stable, asymptotically stable or unstable.

Solution.

(a) We define the required terms:

- A point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is a critical point of $(**)$ if

$$\begin{Bmatrix} x \equiv x_0 \\ y \equiv y_0 \end{Bmatrix}$$

is one of its solutions.

- A critical point

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

of $(**)$ is stable if it satisfies the following condition: for all $r > 0$, there is a neighbourhood $\mathcal{U} \subset \mathbb{R}^2$ of $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ such that every solution $\begin{pmatrix} x \\ y \end{pmatrix}$ of $(**)$ with initial conditions $x(0) = x_1 \in \mathcal{U}$ and $y(0) = y_1 \in \mathcal{U}$ satisfies

$$\forall t \geq 0 : \left\| \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\| < r.$$

- A critical point

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

of $(**)$ is asymptotically stable if it satisfies the following condition: there is a neighbourhood $\mathcal{U} \subset \mathbb{R}^2$ of $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ such that every solution $\begin{pmatrix} x \\ y \end{pmatrix}$ of $(**)$ with initial conditions $x(0) = x_1 \in \mathcal{U}$ and $y(0) = y_1 \in \mathcal{U}$ satisfies

$$\lim_{t\to\infty} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

- Critical points of $(***)$ are exactly those $\begin{pmatrix} x \\ y \end{pmatrix}$ such that

$$\sin(x) + \sin(y) = 0 \qquad \text{and} \qquad (x+1)(y-1) + 1 = 0.$$

The former equation, rewritten as

$$\sin(x) = -\sin(y) = \sin(-y)$$

implies $y \in -x + n\pi$ for some $n \in \pi$. Plugging this into the second equation, we first find that $n = 0$ and then $x = y = 0$.

We investigate stability of the critical point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ by linearisation. The Jacobian of $(***)$ is

$$\begin{pmatrix} \cos(x) & \cos(y) \\ y-1 & x+1 \end{pmatrix}$$

providing us with the linear approximation

$$\begin{pmatrix} \cos(0) & \cos(0) \\ 0-1 & 0+1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The eigenvalues $1+i$ and $1-i$ of the coefficient matrix can be determined by calculating the roots of its characteristic polynomial. Both eigenvalues have a strictly positive real part, so that the critical point is unstable.

4. **(8p)**

   (a) Provide an example of a 2nd order boundary value problem with von Neumann type boundary conditions that does not have a unique solution.

   (b) State the Fredholm alternative for Sturm-Liouville type boundary value problems.

   (c) Rewrite the differential expression $L\,u = u'' - 2u' + u$ in the form of a Sturm-Liouvile problem and find a fundamental solution that is not a Green function.

Solution.

(a) The BVP $u'' = 0$ with von Neumann type boundary conditions $u'(0) = 0 = u'(1)$ is solved by the functions $f_c : [0,1] \to \mathbb{R}$, $f_c(x) \equiv c$ for all $c \in \mathbb{R}$.

(b) The Fredholm alternative for a Sturm-Liouville problem

$$\begin{aligned} Lu &= (p(x)u')' + q(x) \\ R_1 u &= \alpha_1 u(a) + \alpha_2 p(a) u'(a) \\ R_2 u &= \beta_1 u(b) + \beta_2 p(b) u'(b) \end{aligned}$$

says that exactly one of the following two statements is true:

- Either the system

$$\begin{aligned} Lu &= g \\ R_1 u &= \eta_1 \\ R_2 u &= \eta_2 \end{aligned}$$

admits a unique solution for every $g \in \mathrm{C}([a,b])$ and every $\eta_1, \eta_2 \in \mathbb{R}$, or
- the system

$$\begin{aligned} Lu &= 0 \\ R_1 u &= 0 \\ R_2 u &= 0 \end{aligned}$$

admits a non-zero solution.

(c) In order to rewrite the differential expression $u'' - 2u' + u$ in the form of a Sturm-Liouville problem we have to multiply with

$$p(x) = e^{\int_0^x -2ds} = e^{-2x}$$

and obtain

$$e^{-2x}u'' - 2e^{-2x}u' + e^{-2x}u = (e^{-2x}u')' + e^{-2x}u.$$

Say the problem is posed on an interval $[a,b]$. Then a fundamental solution $\Gamma : [a,b] \times [a,b] \to \mathbb{R}$ is given by

$$\Gamma(x,\xi) = \begin{cases} u_\xi(x) & \text{if } a \leq \xi \leq x \leq b \\ 0 & \text{if } a \leq x \leq \xi \leq b \end{cases}$$

where u_ξ is the unique solution to the initial value problem

$$\begin{aligned} Lu &= 0 \\ u(\xi) &= 0 \\ u'(\xi) &= \frac{1}{p(\xi)}. \end{aligned}$$

Let us find these solutions: a fundamental system for $Lu = 0$ is given by e^x and xe^x. So the initial conditions for

$$u_\xi(x) = ae^x + bxe^x$$

give rise to the equations

$$\begin{aligned} ae^\xi + b\xi e^\xi &= 0 \\ ae^\xi + b\xi e^\xi + be^\xi &= e^{2\xi}. \end{aligned}$$

Working out these conditions, we obtain

$$\begin{aligned} a &= -\xi e^\xi \\ b &= e^\xi. \end{aligned}$$

So a fundamental solution of the Sturm-Liouville problem is

$$\Gamma(x,\xi) = \begin{cases} -\xi e^\xi e^x + e^\xi x e^x & \text{if } a \leq \xi \leq x \leq b \\ 0 & \text{if } a \leq x \leq \xi \leq b \end{cases}$$

This fundamental solution is not a Green function, since it is not symmetric.