MATEMATISKA INSTITUTIONEN	Exam in
STOCKHOLMS UNIVERSITET	Ordinary differential equations
Avd. Matematik	7.5 hp
Examiner: Sven Raum	25th May 2020
Instructor: Corentin Léna	9:00 to 16:00

Please read carefully the general instructions:

- This is an open book exam.
- In all your solutions show your reasoning and calculation, explaining carefully what you are doing. Justify your answers.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- Upload the exam on the course page before 16 o'clock
- A maximum score of 100 points can be achieved. A score of at least 50 points will ensure a pass grade if item 0 is completed.

GOOD LUCK!

0. **Mandatory, but gives no points.**
The PDF document that contains your home exam should start by you writing the following sentence:

   I, the author of this document, hereby guarantee that I have produced these solutions to this home exam without the assistance of any other person. This means that I have for example not discussed the solutions or the home exam with any other person.

1. **Systems of differential equations** (20 points)
Find the general solution of the system of differential equations

$$X' = AX$$

   where A is the matrix

$$\begin{pmatrix} 7 & 1 & \sqrt{2} \\ 1 & 7 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 6 \end{pmatrix}.$$

2. **Higher order differential equations** (20 points)
Solve the differential equation

$$f''(x) - f'(x) - 2f(x) = 6xe^{-x}$$
$$f(0) = 0$$
$$f'(0) = -\frac{2}{3}.$$

3. **Power series method** (20 points)
Solve the following differential equation by means of the power series method and express its solution as an elementary function.

$$-x(x+1)^2 f'(x) + (x+1)^2 f(x) = x^2$$
$$f(0) = 0$$
$$f(1) = \frac{1}{2}.$$

4. **Autonomous systems of differential equations** (20 points)
For the following autonomous system, find all equilibrium points and determine whether they are asymptotically stable, stable or unstable

$$\begin{Bmatrix} x' = \sin x \cdot \cos y \\ y' = x + y \end{Bmatrix}$$

5. **Boundary value problems** (20 points)
Consider the differential equation

$$u'' - 2xu' + 2nu = 0 \tag{$*$}$$

   for a parameter $n \in \mathbb{N}$.

   (a) Rewrite the differential equation in Sturm-Liouville form.

   (b) Find a solution H_0 for the boundary value problem

$$u'' - 2xu' = 0,$$
$$H_0'(0) = 0 = H_0'(1).$$

   (c) Show that if H_n is a solution of $(*)$ for the parameters n, then there is a solution H_{n-1} for the parameter $n-1$ that satisfies $H_n' = nH_{n-1}$.

   (d) Use the statement of the previous item to find solutions H_1, H_2, H_3, H_4 for the differential equation $(*)$ with parameters $n = 1, 2, 3, 4$.